Dynamic Meteorology 1

Lecture 3

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Equations of motion on rotating earth

In scalar form the equations of motion for each direction become:

(East-West equation)

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + K\frac{\partial^2 u}{\partial z^2} + 2\Omega v\sin\phi - 2\Omega w\cos\phi$$

(North-South equation)

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + K\frac{\partial^2 v}{\partial z^2} - 2\Omega u\sin\phi$$

(Up-Down equation)

$$\frac{dw}{dt} - \frac{u^2 + v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega u\cos\phi$$

Acceleration

Correction terms for spherical coordinate system

Pressure gradient Force

Friction

Effective Gravity

Coriolis Force

The complete momentum equations on a spherical rotating earth

$$\frac{du}{dt}-\frac{uv\tan\phi}{a}+\frac{uw}{a}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+K\frac{\partial^2 u}{\partial z^2}+2\Omega v\sin\phi-2\Omega w\cos\phi$$

$$\frac{dv}{dt}+\frac{u^2\tan\phi}{a}+\frac{vw}{a}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+K\frac{\partial^2 v}{\partial z^2}-2\Omega u\sin\phi$$

$$\frac{dw}{dt}-\frac{u^2+v^2}{a}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g+2\Omega u\cos\phi$$

FOR SYNOPTIC SCALE MOTIONS.....

WHICH TERMS ARE LARGE AND IMPORTANT?
WHICH TERMS ARE SMALL AND INSIGNIFICANT?

$$\frac{\partial u}{\partial t}=-u\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial y}-w\frac{\partial u}{\partial z}+\frac{uv\tan\varphi}{a}-\frac{uw}{a}-\frac{1}{\rho}\frac{\partial p}{\partial x}-2\Omega(w\cos\varphi-v\sin\varphi)+F_{rx}$$

$$\frac{\partial v}{\partial t}=-u\frac{\partial v}{\partial x}-v\frac{\partial v}{\partial y}-w\frac{\partial v}{\partial z}-\frac{u^2\tan\varphi}{a}-\frac{vw}{a}-\frac{1}{\rho}\frac{\partial p}{\partial y}-2\Omega u\sin\varphi+F_{ry}$$

$$\frac{\partial w}{\partial t}=-u\frac{\partial w}{\partial x}-v\frac{\partial w}{\partial y}-w\frac{\partial w}{\partial z}-\frac{u^2+v^2}{a}-\frac{1}{\rho}\frac{\partial p}{\partial z}+2\Omega u\cos\varphi-g+F_{rz}$$

$$\frac{\partial T}{\partial t}=-u\frac{\partial T}{\partial x}-v\frac{\partial T}{\partial y}+(\gamma-\gamma_d)w+\frac{1}{a}-\frac{1}{c_p}\frac{dH}{dt}$$

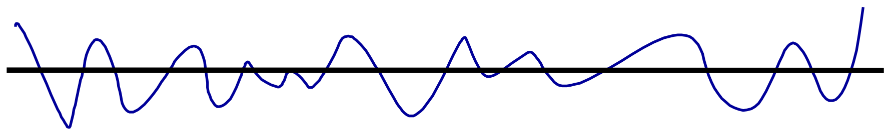
$$\frac{\partial \rho}{\partial t}=-u\frac{\partial \rho}{\partial x}-v\frac{\partial \rho}{\partial y}-w\frac{\partial \rho}{\partial z}-\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

$$\frac{\partial q_v}{\partial t}=-u\frac{\partial q_v}{\partial x}-v\frac{\partial q_v}{\partial y}-w\frac{\partial q_v}{\partial z}+Q_v$$

$$p=\rho RT$$

Next, we will assume that the density field may be expressed as a sum of a referenced density which is function only of height and a small perturbation, that is:

$$\rho = \bar{\rho}(z) + \rho'(x, y, z, t)$$



$$u = \bar{u} + u',$$

$$T = \bar{T} + T'$$

$$p = \bar{p} + p'.$$

$$u\frac{\partial u}{\partial x} = (\bar{u}+u')\frac{\partial}{\partial x}(\bar{u}+u') = \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{u}\frac{\partial u'}{\partial x} + u'\frac{\partial \bar{u}}{\partial x} + u'\frac{\partial u'}{\partial x}$$

$$\overline{u\frac{\partial u}{\partial x}} = \overline{\bar{u}\frac{\partial \bar{u}}{\partial x}} + \overline{\bar{u}\frac{\partial u'}{\partial x}} + \overline{u'\frac{\partial \bar{u}}{\partial x}} + \overline{u'\frac{\partial u'}{\partial x}}$$

$$\overline{a'} = 0, \qquad \bar{\bar{a}} = \bar{a}$$

$$\overline{\bar{a}b} = \bar{\bar{a}}\bar{\bar{b}} = \bar{a}\bar{b}$$

$$\overline{\bar{a}b'} = \bar{\bar{a}}\overline{b'} = \bar{a}\overline{b'} = 0$$

$$\overline{u\frac{\partial u}{\partial x}} = \bar{u}\frac{\partial \bar{u}}{\partial x} + \cancelto{0}{\bar{u}\overline{\frac{\partial u'}{\partial x}}} + \cancelto{0}{\overline{u'}\frac{\partial \bar{u}}{\partial x}} + \overline{u'\frac{\partial u'}{\partial x}} = \bar{u}\frac{\partial \bar{u}}{\partial x} + \overline{u'\frac{\partial u'}{\partial x}}.$$

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial x} + fv + \frac{1}{\rho}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) \quad *$$

$$\tau_{zx} = \mu\frac{\partial u}{\partial z}$$

$$\frac{\partial u}{\partial t} \propto \frac{1}{\rho}\left(\mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial^2 u}{\partial z^2}\right) = \frac{\mu}{\rho}\nabla^2 u$$

$$\frac{\partial \bar{u}}{\partial t} = -\bar{u}\frac{\partial \bar{u}}{\partial x} - \bar{v}\frac{\partial \bar{u}}{\partial y} - \bar{w}\frac{\partial \bar{u}}{\partial z} - \frac{1}{\bar{\bar{\rho}}}\frac{\partial \bar{p}}{\partial x} + f\bar{v} - \overline{u'\frac{\partial u'}{\partial x}} - \overline{v'\frac{\partial u'}{\partial y}} - \overline{w'\frac{\partial u'}{\partial z}} + \frac{1}{\bar{\bar{\rho}}}\left(\frac{\partial \bar{\tau}_{xx}}{\partial x} + \frac{\partial \bar{\tau}_{yx}}{\partial y} + \frac{\partial \bar{\tau}_{zx}}{\partial z}\right)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

$$\frac{\partial \bar{u}}{\partial t} = -\bar{u}\frac{\partial \bar{u}}{\partial x} - \bar{v}\frac{\partial \bar{u}}{\partial y} - \bar{w}\frac{\partial \bar{u}}{\partial z} - \frac{1}{\bar{\bar{\rho}}}\frac{\partial \bar{p}}{\partial x} + f\bar{v} - \frac{\overline{\partial u'u'}}{\partial x} - \frac{\overline{\partial u'v'}}{\partial y} - \frac{\overline{\partial u'w'}}{\partial z} + \frac{1}{\bar{\bar{\rho}}}\left(\frac{\partial \bar{\tau}_{xx}}{\partial x} + \frac{\partial \bar{\tau}_{yx}}{\partial y} + \frac{\partial \bar{\tau}_{zx}}{\partial z}\right). *$$

$$T_{xx} = -\bar{\rho}\,\underline{\overline{u'u'}}$$

$$T_{yx} = -\bar{\rho}\,\overline{u'v'}$$

$$T_{zx} = -\bar{\rho}\;\overline{u'w'}.$$

$$\frac{\partial \bar{u}}{\partial t} = -\bar{u}\frac{\partial \bar{u}}{\partial x} - \bar{v}\frac{\partial \bar{u}}{\partial y} - \bar{w}\frac{\partial \bar{u}}{\partial z} - \frac{1}{\bar{\bar{\rho}}}\frac{\partial \bar{p}}{\partial x} + f\bar{v}$$

$$+ \frac{1}{\bar{\bar{\rho}}}\left(\frac{\partial}{\partial x}(\tau_{xx} + T_{xx}) + \frac{\partial}{\partial y}(\tau_{yx} + T_{yx}) + \frac{\partial}{\partial z}(\tau_{zx} + T_{zx})\right)$$

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial x} + fv + \frac{1}{\rho}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) \quad *$$

Scale Analysis of the Equations of Motion

Scale analysis, or scaling, is a convenient technique for estimating the magnitudes of various terms in the governing equations for a particular type of motion.

Elimination of terms on scaling considerations simplify mathematics and allows to eliminate and filter unwanted types of motions, like sound waves.

In order to simplify the motion equations for synoptic scale motions we define following characterstic scales of the field variables based on observed valuses for midlatitude synoptic systems.

$U \sim$	$10\ m\ s^{-1}$	horizontal velocity scale
$W \sim$	$0.01\ m\ s^{-1}$	vertical velocity scale
$L \sim$	$10^6\ m$	horizontal length scale
$H \sim$	$10^4\ m$	vertical length scale
$p \sim$	$1000\ hPa\ (10^5\ Pa)$	mean surface pressure
$\delta P/\rho \sim$	$10\ hPa/1 \sim (10^3\ m^2 s^{-2})$	pressure horizontal variation scale
$L/U \sim 10^5\ s$		time scale

$d/dt = 1/\Delta t \sim U/L \sim 10^{-5}\ s^{-1}$

Scale Analysis of the Horizontal Momentum Equation

It is possible to scale each term of each horizontal equation using previously defined scales and the definition of the Coriolis parameter:

$$\Omega = 7.27 \times 10^{-5}\ s^{-1}$$

$$f_0 = 2\Omega \sin\varphi_0 = 2\Omega \cos\varphi_0 = 10^{-4} s^{-1}$$

$$x-eq. \quad \frac{du}{dt}-\frac{uv\tan\phi}{a}+\frac{uw}{a}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+2\Omega v\sin\phi-2\Omega w\cos\phi+F_{rx}$$

$$y-eq. \quad \frac{dv}{dt}+\frac{u^2\tan\phi}{a}+\frac{vw}{a}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-2\Omega u\sin\phi \qquad +F_{ry}$$

Scales	$\frac{U^2 \approx 10^2}{L \approx 10^6}$	$\frac{U^2 = 10^2}{a = 10^7}$	$\frac{UW}{a}$	$\frac{\delta P}{\rho L}$	$f_0 U =$	$f_0 W$	$\frac{\nu U}{H^2}$
(ms^{-2})	10^{-4}	10^{-5}	10^{-8}	10^{-3}	10^{-3}	10^{-6}	10^{-12}

The Geostrophic Approximation and Geostrophic wind

Retaining only two bigger terms from the analysis equation, it is possible to write the first approximation geostrophic balance:

$$- fv \approx -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad fu \approx -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\vec{\mathbf{V}}_g \equiv \hat{\mathbf{k}} \times \frac{1}{\rho f}\nabla p$$

This equation is diagnostic (no reference to time) and allows the definition of geostrophic wind as balance between Coriolis and pressure gradient forces.

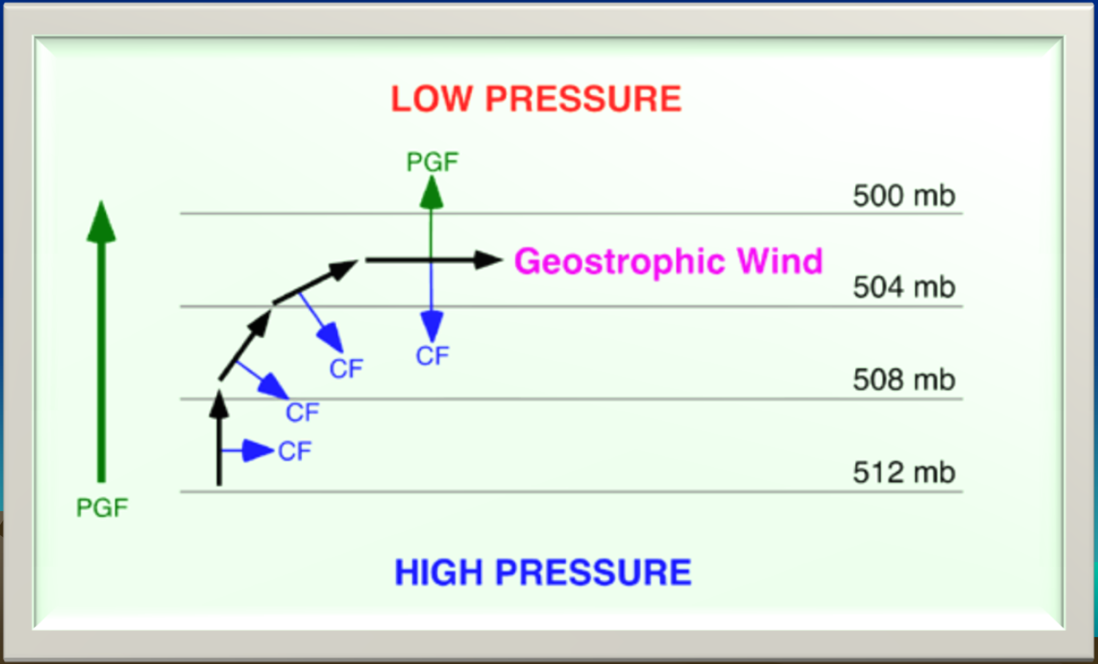
Geostrophic Wind

Winds aloft (above ~1000 m) flowing in a straight line, a balance between 2 forces:

Pressure gradient force (PGF)

Coriolis 'force' (CF)

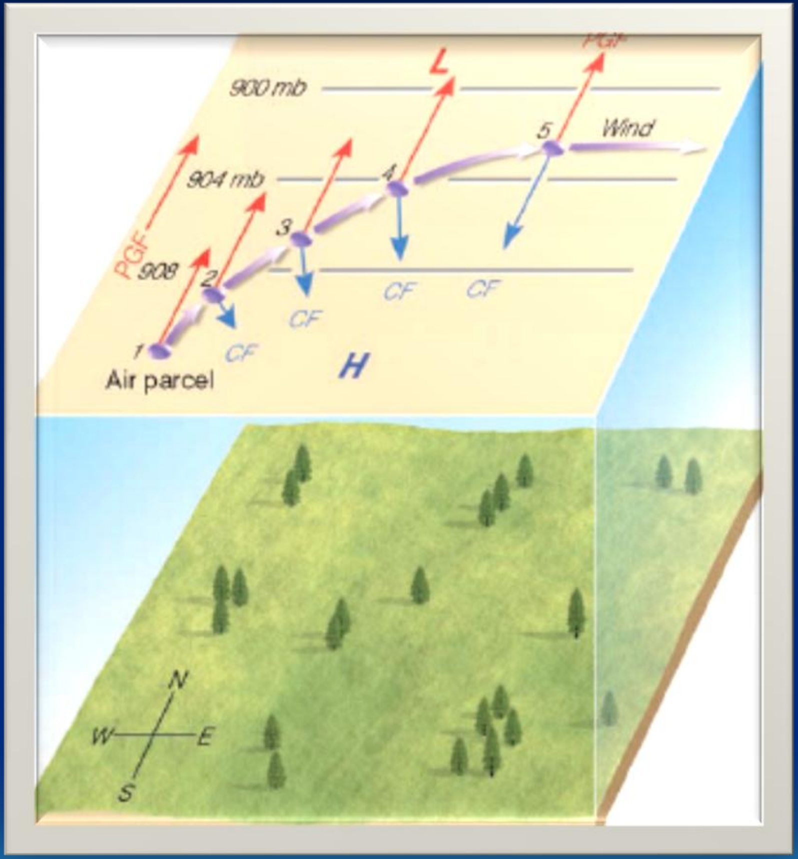
A wind that begins to blow across the isobars is turned by the Coriolis 'force' until Coriolis 'force' and PGF balance



Geostrophic Wind

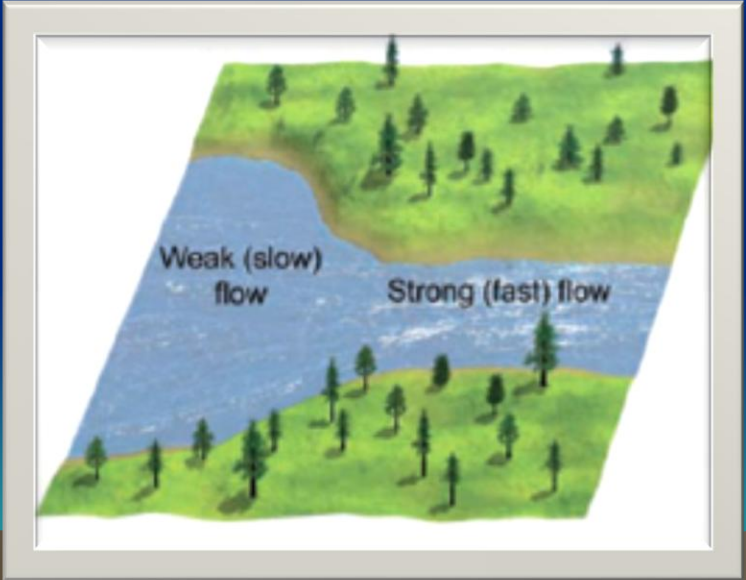
Above the level of friction, air initially at rest will accelerate until it flows parallel to the isobars at a steady speed with the pressure gradient force (PGF) balanced by the Coriolis force (CF).

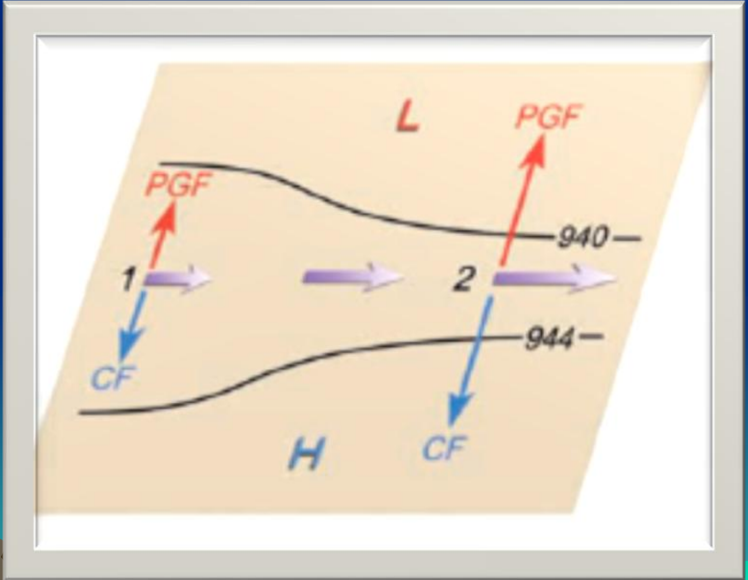
Wind blowing under these conditions is called geostrophic.



The isobars and contours on an upper-level chart are like the banks along a flowing stream. When they are widely spaced, the flow is weak; when they are narrowly spaced, the flow is stronger.

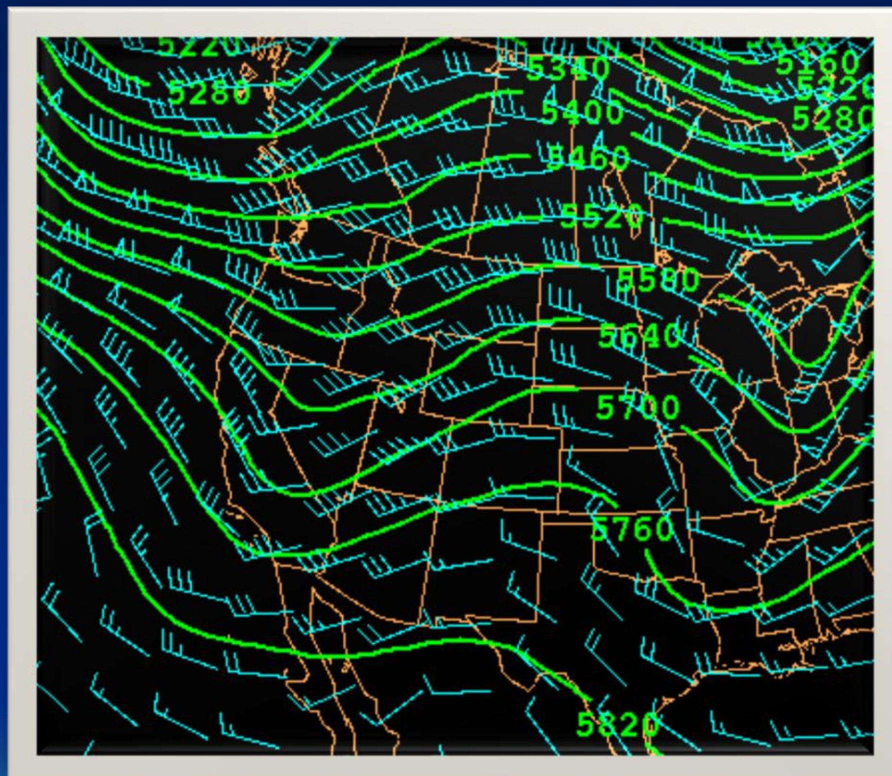
The increase in winds on the chart results in a stronger Coriolis force (CF), which balances a larger pressure gradient force (PGF).





Geostrophic Flow at 500 mb

When the flow is parallel to approximately straight height lines, the flow is geostrophic.



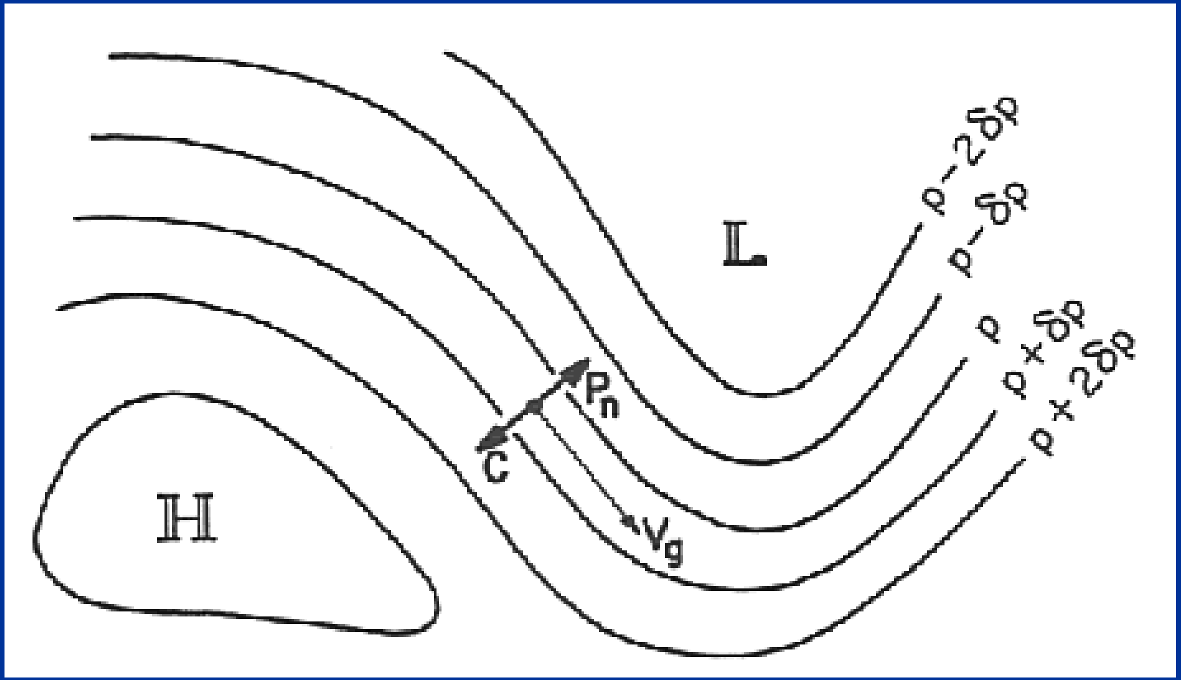
what determines the strength of the geostrophic flow?

The magnitudes of The Pressure Gradient Force

Geostrophic wind

In the absence of any other “forces”, the Coriolis force balances the PGF and the flow is steady.

This is called the **Geostrophic Wind**. On a weather map, say at 500 mb, the wind vectors are usually parallel to the contours, and the flow around a cyclone is anticlockwise in the NH.



Approximate Prognostic Equation

To obtain prediction equation it is necessary to retain also acceleration term. The resulting approximate horizontal equations can be written as:

$$\frac{du}{dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} = f\left(v - v_g\right)$$

$$\frac{dv}{dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} = -f\left(u - u_g\right)$$

Since acceleration terms in above equations are proportional to the difference between the actual wind and the geostrophic wind,

The Rossby Number

The Rossby Number is a dimensionaless number used in describing geophysical phenomena in the oceans and atposphere.

It characterises the ratio ofinertial forces in a fluid to the fictitious forces arising from planetary rotation.

$$R_o \equiv \frac{U^2 / L}{f_0 U} = \frac{U}{f_0 L}$$

When the Rossby number is large (such as in the tropics and at lower latitudes), the effects of Planetary rotation are unimportant and can be neglected.

When the Rossby number is small ($R_0 \ll 1$) , then the effects of planetary rotation are large and the geostrophic approximation is valid.

The Hydrostatic Approximation

A similar scale analysis can be applied to the vertical Component of the momentum equation.

$$z-eq. \qquad \frac{dw}{dt} - \frac{u^2+v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + 2\Omega u\cos\phi + g + F_{rz}$$

$Scales$	UW/L	$\frac{U^2}{a}$	$\frac{P_0}{\rho H}$	$f_0 U$	g	$\frac{\nu W}{H^2}$
(ms^{-2})	10^{-7}	10^{-5}	10	10^{-3}	10	10^{-15}

The biggest terms give the hydrostatic approximation:

$$\frac{1}{\rho_0}\frac{dp_0}{dz} \equiv -g$$

$P_0(z)$ is the horizontally averaged pressure at each height.

$\rho_0(z)$ is standard density.

We may then write the total pressure and density fields as:

$$P(x, y, z, t) = P_0(z) + P'(x, y, z, t)$$

$$\rho(x, y, z, t) = \rho_0(z) + \rho'(x, y, z, t)$$

Where P′ and ρ′ are deviation from the standard values of pressure and density (perturbation). For an atmosphere at rest, would thus be zero.

$$-\frac{1}{\rho}\frac{\partial P}{\partial z} - g = -\frac{1}{\rho_0 + \rho'}\frac{\partial}{\partial z}(P_0 + P') - g$$

$$(\rho_0+\rho')^{-1}=\rho_0^{-1}(1+\rho'/\rho_0)^{-1}\cong\rho_0^{-1}(1-\rho'/\rho_0)$$

$$\rho_0^{-1}(1-\rho'/\rho_0)(\partial P_0/\partial z+\partial P'/\partial z)+g=$$

$$\rho_0^{-1}\left(\frac{\partial P_0}{\partial z}+\frac{\partial P'}{\partial z}-\frac{\rho'}{\rho_0}\frac{\partial P_0}{\partial z}-\frac{\rho'}{\rho_0}\frac{\partial P'}{\partial z}\right)+g$$

$$-g \qquad \approx\frac{\delta P}{\rho_0 H}\approx 10^{-1}ms^{-2} \qquad (\rho'/\rho_0)g=10^{-1} \qquad 10^{-3}$$

$$\frac{1}{\rho_0}\frac{\partial P'}{\partial z}+\frac{\rho'}{\rho_0}g=0 \qquad \frac{1}{\rho_0}\left(\rho' g+\frac{\partial P'}{\partial z}\right)=0 \qquad \frac{1}{\rho'}\frac{\partial P'}{\partial z}=-g$$

نتیجه گیری: تقریب هیدروستاتیک برای پریشیدگی افقی نیز صدق **می کند**.

ABOVE THE BOUNDARY LAYER

$$\frac{du}{dt} - \frac{uv\tan\varphi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + K\frac{\partial^2 u}{\partial z^2} \qquad + 2\Omega v\sin\varphi - 2\Omega w\cos\varphi$$

$$\frac{dv}{dt} + \frac{u^2\tan\varphi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + K\frac{\partial^2 v}{\partial z^2} \qquad - 2\Omega u\sin\varphi$$

$$\frac{dw}{dt} - \frac{u^2+v^2}{a} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + 0 - g + 2\Omega u\cos\varphi$$

WITHIN THE BOUNDARY LAYER

$$\frac{du}{dt} - \frac{uv\tan\varphi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + K\frac{\partial^2 u}{\partial z^2} \qquad + 2\Omega v\sin\varphi - 2\Omega w\cos\varphi$$

$$\frac{dv}{dt} + \frac{u^2\tan\varphi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + K\frac{\partial^2 v}{\partial z^2} \qquad - 2\Omega u\sin\varphi$$

$$\frac{dw}{dt} - \frac{u^2+v^2}{a} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + 0 - g + 2\Omega u\cos\varphi$$

The Hidden Simplicity of Atmospheric Dynamics:

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + fv$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - fu$$

ABOVE THE BOUNDARY LAYER, ALL HORIZONTAL PARCEL ACCELERATIONS CAN BE UNDERSTOOD BY COMPARING THE MAGNITUDE AND DIRECTION OF THE PRESSURE GRADIENT AND CORIOLIS FORCES

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + K\frac{\partial^2 u}{\partial z^2} + fv$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + K\frac{\partial^2 v}{\partial z^2} - fu$$

WITHIN THE BOUNDARY LAYER, ALL HORIZONTAL PARCEL ACCELERATIONS CAN BE UNDERSTOOD BY COMPARING THE MAGNITUDE AND DIRECTION OF THE PRESSURE GRADIENT, CORIOLIS AND FRICTIONAL FORCES

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g$$

THE ATMOSPHERE IS IN HYDROSTATIC BALANCE – VERTICAL PGF BALANCES GRAVITY – ON SYNOPTIC SCALES