Meteorology

Lecture 10

Sahraei

Physics Department

Razi University

http://www.razi.ac.ir/sahraei

Quasi-Static Models

Vertical differencing is a very different problem from horizontal differencing.

This may seem odd, but the explanation is very simple. There are three primary factors.

First, gravitational effects are very powerful, and act only in the vertical.

Second, the Earth's atmosphere is very shallow compared to its horizontal extent.

Third, the atmosphere has a lower boundary.

$$\frac{\partial u}{\partial t} = -\vec{V}.\nabla u - \frac{1}{\rho}\frac{\partial p}{\partial x} + fv - ew + F_x$$

$$\frac{\partial v}{\partial t} = -\vec{V}.\nabla v - \frac{1}{\rho}\frac{\partial p}{\partial y} - fu + F_y$$

$$\frac{\partial w}{\partial t} = -\vec{V}.\nabla w - \frac{1}{\rho}\frac{\partial p}{\partial z} + eu - g + F_z \quad \rightarrow -\frac{1}{\rho}\frac{\partial p}{\partial z} = g$$

$$\frac{\partial p}{\partial t} = -\vec{V}.\nabla p - \gamma p \nabla.\vec{V} + \frac{R}{c_v}\rho\frac{dH}{dt}$$

$$\frac{\partial \rho}{\partial t} = -\vec{V}.\nabla \rho - \rho\nabla.\vec{V}$$

$$\frac{\partial q_v}{\partial t} = -\vec{V}.\nabla q_v + Q_v$$

$$p = \rho RT$$

Calculating vertical motion using Richardson's equation

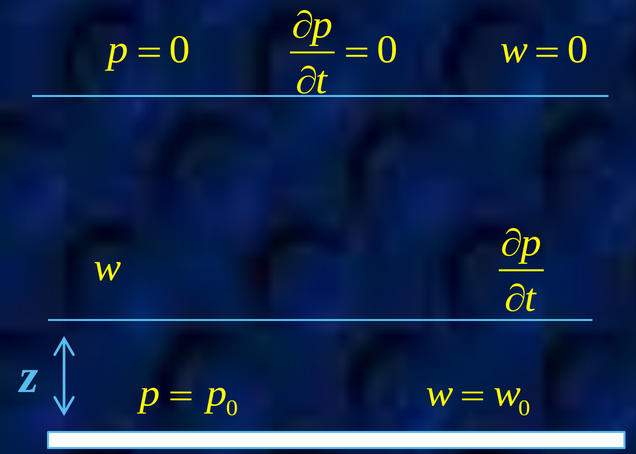
$$\frac{\partial p}{\partial z} = -\rho g \qquad \frac{\partial}{\partial t}\left[\frac{\partial p}{\partial z} = -\rho g\right] \qquad \frac{\partial}{\partial z}\left(\frac{\partial p}{\partial t}\right) = -g\frac{\partial \rho}{\partial t}$$

$$\frac{\partial \rho}{\partial t} = -\vec{V}.\nabla\rho - \rho\nabla.\vec{V} = -\nabla.(\rho\vec{V})$$

$$\frac{\partial \rho}{\partial t} = -\nabla_H.(\rho\vec{U}) - \frac{\partial(\rho w)}{\partial z}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial t}\right) = g\nabla_H.(\rho\vec{U}) + g\frac{\partial(\rho w)}{\partial z}$$

$$\left.\frac{\partial p}{\partial t}\right|_0^{\frac{\partial p}{\partial t}} = g\int_\infty^z \nabla_H.(\rho\vec{U})dz + \rho g w$$



$$\begin{cases} \dfrac{\partial p}{\partial t} = g\displaystyle\int_\infty^z \nabla_H.(\rho\vec{U})dz + \rho g w \\ \dfrac{\partial p}{\partial t} = -\vec{V}.\nabla p - \gamma p \nabla.\vec{V} + \dfrac{R}{c_v}\rho\dfrac{dH}{dt} \end{cases}$$

$$\frac{\partial p}{\partial t} = -\vec{U}.\nabla p - w\frac{\partial p}{\partial z} - \gamma p\nabla.\vec{U} - \gamma p\frac{\partial w}{\partial z} + \frac{R}{c_v}\rho\frac{dH}{dt}$$

$$g\int_\infty^z \nabla_H.(\rho\vec{U})dz + \rho \cancel{g} w = -\vec{U}.\nabla p + w\cancel{\rho} g - \gamma p\nabla.\vec{U} - \gamma p\frac{\partial w}{\partial z} + \frac{R}{c_v}\rho\frac{dH}{dt}$$

$$\frac{1}{\gamma p}\left(g\int_\infty^z \nabla_H.(\rho\vec{U})dz + \vec{U}.\nabla p\right) = -\nabla.\vec{U} - \frac{\partial w}{\partial z} + \frac{R}{\gamma p c_v}\rho\frac{dH}{dt}$$

$$\frac{\partial w}{\partial z} = -\nabla.\vec{U} + \frac{1}{\gamma p}\left[-\vec{U}.\nabla p + \frac{R}{c_v}\rho\frac{dH}{dt} - g\int_{\infty}^{z}\nabla_H.(\rho\vec{U})dz \right]$$

$$\frac{\partial w}{\partial z} = A \qquad A = A(x, y, z, t)$$

$$w_R = w_0 + \int_0^z A(x, y, z, t)dz$$

Quasi-Static equations System in Richardson's Method

$$\frac{\partial u}{\partial t} = -\vec{V}.\nabla u - \frac{1}{\rho}\frac{\partial p}{\partial x} + fv + F_x$$

$$p = \rho RT$$

$$\frac{\partial v}{\partial t} = -\vec{V}.\nabla v - \frac{1}{\rho}\frac{\partial p}{\partial y} - fu + F_y$$

$$w_R = w_0 + \int_0^z A(x, y, z, t)dz$$

$$\frac{\partial p}{\partial t} = -\vec{V}.\nabla p - \gamma p \nabla.\vec{V} + \frac{R}{c_v}\rho\frac{dH}{dt}$$

$$\rho = -\frac{1}{g}\frac{\partial p}{\partial z}$$

Prediction in P Coordinate

$$\frac{du}{dt} = -\frac{\partial \phi}{\partial x} + fv + \frac{1}{\rho} F_x$$

$$p\alpha = RT$$

$$\frac{dv}{dt} = -\frac{\partial \phi}{\partial y} - fu + \frac{1}{\rho} F_y$$

$$\theta = T(\frac{p_0}{p})^k$$

$$\frac{\partial \phi}{\partial p} = -\alpha$$

$$\frac{\partial \phi}{\partial p} = -\frac{R}{p}(\frac{p}{p_0})^k \theta \quad = -\hat{R}\theta$$

$$(\frac{\partial u}{\partial x})_p + (\frac{\partial u}{\partial y})_p + \frac{\partial \omega}{\partial p} = 0$$

$$\hat{R} = \frac{R}{p}(\frac{p}{p_0})^k = \frac{\alpha}{\theta}$$

$$\frac{d \ln \theta}{dt} = \frac{1}{c_p T} \dot{H}$$

$$\frac{\partial \phi}{\partial p} = -\alpha = -\hat{R}\theta$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}+\omega\frac{\partial u}{\partial p}=-\frac{\partial \phi}{\partial x}+fv+f_x \quad (1)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\omega\frac{\partial v}{\partial p}=-\frac{\partial \phi}{\partial y}-fu+f_y \quad (2)$$

$$\frac{\partial \phi}{\partial p}=-\hat{R}\theta \quad (3)$$

$$\left.\begin{cases} t_0: \quad u_0, v_0, \theta_0 \\ t=t_0+\delta t: \quad u, v, \theta \end{cases}\right\} \Rightarrow (3),(4) \Rightarrow \phi, \omega$$

$$\frac{\partial \omega}{\partial p}=-\nabla_p . \vec{V}_H \quad (4)$$

$$\frac{\partial \theta}{\partial t}+u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}+\omega\frac{\partial \theta}{\partial p}=\frac{\theta}{c_p T}\dot{H}=\frac{1}{c_p}\left(\frac{p_0}{p}\right)^k \dot{H} \quad (5)$$

$$\frac{d \ln \theta}{dt}=\frac{1}{\theta}\frac{d\theta}{dt}$$

(a) (b)

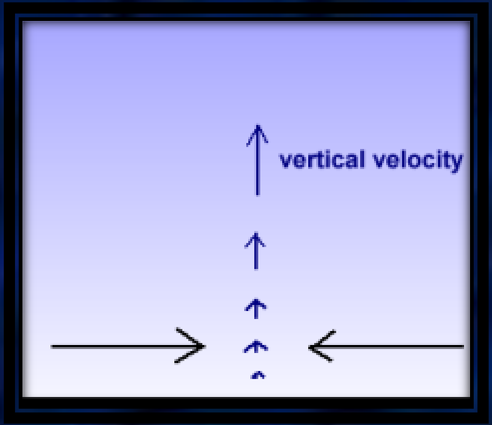
$$\frac{\partial \omega}{\partial p} = -\nabla_p . \vec{V}_H \quad (4)$$

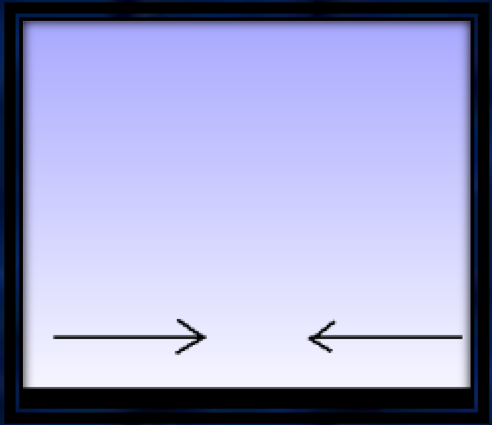
where (a) is the rate of change of vertical velocity with pressure level and (b) is the horizontal convergence.

This is a very important relationship.

It says that the vertical velocity and horizontal convergence fields are intimately linked.

A convergence or divergence of mass will induce vertical airflow in the atmosphere.





Boundary Conditions

1) $\forall(t) \quad , \quad \omega=0 \quad , \quad p=0$

2) $\omega_s = (\frac{dp}{dt})_s = (\frac{\partial p}{\partial t})_z + \vec{V}_H . \nabla_z p + w\frac{\partial p}{\partial z}$

$$\omega_s = (\frac{dp}{dt})_s = (\frac{\partial p}{\partial t})_s + \vec{V}_{sH} . \nabla_s p + w\frac{\partial p}{\partial z}$$

$$\omega_s = (\frac{dp}{dt})_s = (\frac{\partial p}{\partial t})_s + \vec{V}_{sH} . \nabla_s p$$

$$\omega_s = (\frac{\partial p}{\partial t})_s + \vec{V}_{sH} . \nabla_s p$$

$$\omega_s = \frac{\partial p_s}{\partial t} + \vec{V}_{sH} . \nabla p_s \quad (4)$$

$$\frac{\partial \omega}{\partial p} = -\nabla_p . \vec{V}_H \quad (4)$$

$$\omega_s = -\int_0^{p_s} \nabla_p . \vec{V}_H \, dp \quad (6)$$

$$p_s = p_s(x, y, t)$$

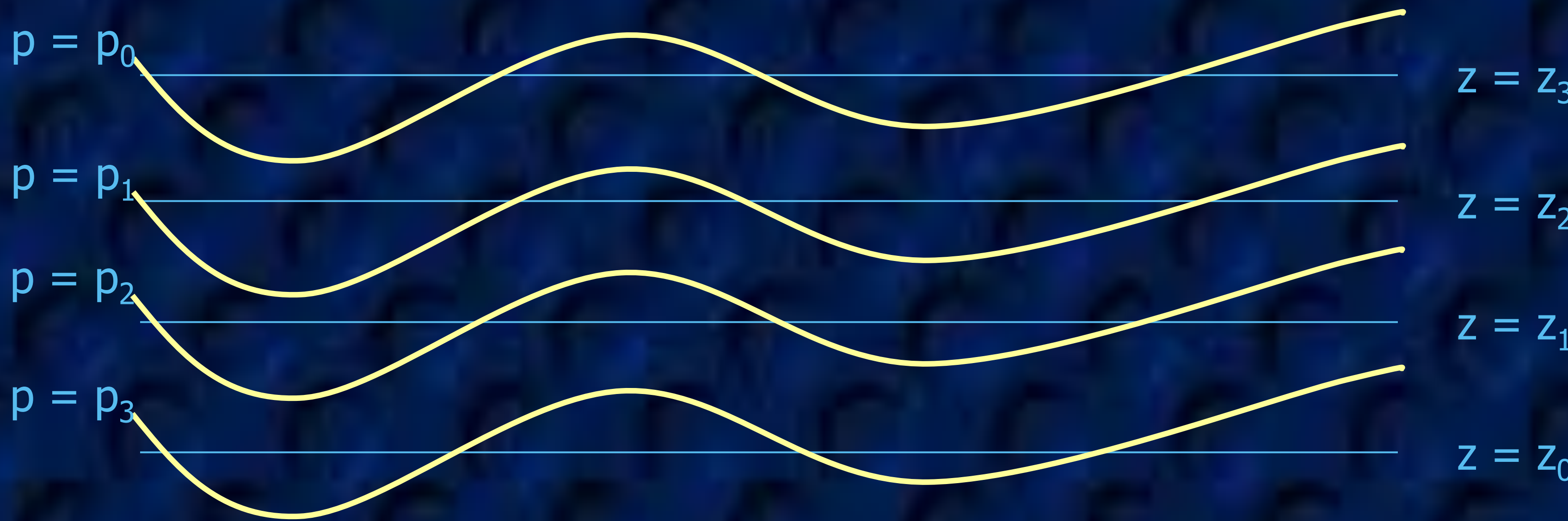
$$(4) = (6) \quad \rightarrow \frac{\partial p_s}{\partial t} = -\int_0^{p_s} \nabla_p . \vec{V}_H \, dp \; - \vec{V}_{sH} . \nabla p_s \quad (7)$$

$$\frac{dp}{dt} = \left(\frac{\partial p}{\partial t}\right)_p + \vec{V} . \nabla_p p + \omega \frac{\partial p}{\partial p} = \omega$$

$$\omega = \frac{dp}{dt} = -\rho g w$$

$$\frac{dp}{dt} = \left(\frac{\partial p}{\partial t}\right)_z + \vec{V} . \nabla_z p + w \frac{\partial p}{\partial z} = -\rho g w$$

Pressure As A Vertical Coordinate



$p_3 > p_2 > p_1 > p_0$

$z_3 > z_2 > z_1 > z_0$

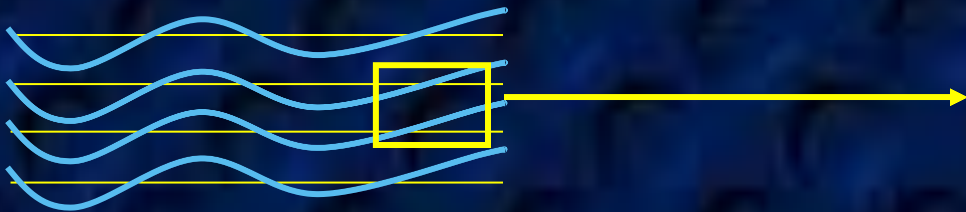
How do we convert our equations from height coordinates (x,y,z) to pressure coordinates (x,y,p)?

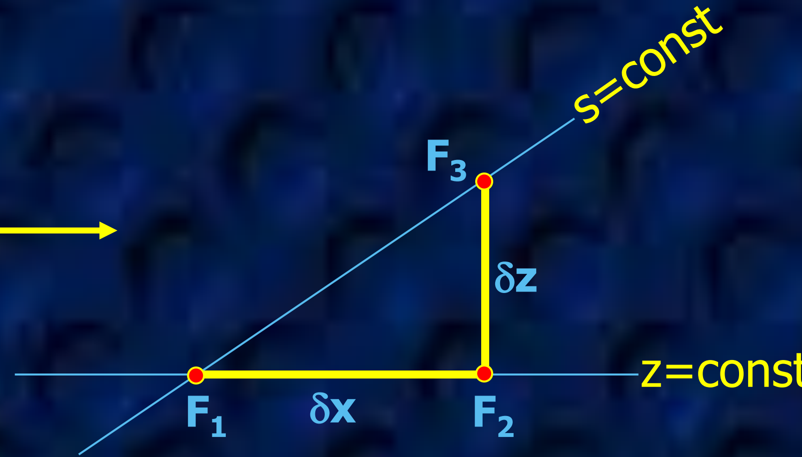
Generalized Vertical Coordinates

The use of pressure as a vertical coordinate is a specific example of the use of generalized vertical coordinates.

Any quantity $s = s(x,y,z,t)$ that changes monotonically with height can be used as a vertical coordinate.

If we wish to transform equations from (x,y,z) coordinates to (x,y,s) coordinates, derivatives must be transformed.





Let F = some scalar property, and
s = a generalized vertical coordinate.

We would like to transform derivatives such as

$$\left(\frac{\partial F}{\partial x}\right)_z \text{ to } \left(\frac{\partial F}{\partial x}\right)_s$$

Derivative in x-direction on a constant z surface

Derivative in x-direction on a constant s surface

$$\frac{F_3 - F_1}{\delta x} = \frac{F_2 - F_1}{\delta x} + \frac{F_3 - F_2}{\delta x} = \frac{F_2 - F_1}{\delta x} + \frac{F_3 - F_2}{\delta z}\frac{\delta z}{\delta x}$$

$$\left(\frac{\delta F}{\delta x}\right)_s = \left(\frac{\delta F}{\delta x}\right)_z + \frac{\delta F}{\delta z}\left(\frac{\delta z}{\delta x}\right)_s$$

$$\begin{cases} \left(\frac{\partial F}{\partial x}\right)_s = \left(\frac{\partial F}{\partial x}\right)_z + \frac{\partial F}{\partial z}\left(\frac{\partial z}{\partial x}\right)_s \\ \left(\frac{\partial F}{\partial y}\right)_s = \left(\frac{\partial F}{\partial y}\right)_z + \frac{\partial F}{\partial z}\left(\frac{\partial z}{\partial y}\right)_s \end{cases}$$

$$\nabla_s F = \left(\frac{\partial F}{\partial x}\right)_s \hat{i} + \left(\frac{\partial F}{\partial y}\right)_s \hat{j}$$

$$\nabla_z F = \left(\frac{\partial F}{\partial x}\right)_z \hat{i} + \left(\frac{\partial F}{\partial y}\right)_z \hat{j}$$

$$\nabla_s z = \left(\frac{\partial z}{\partial x}\right)_s \hat{i} + \left(\frac{\partial z}{\partial y}\right)_s \hat{j}$$

can be written in vector form as

$$\nabla_s F = \nabla_z F + \frac{\partial F}{\partial z}\nabla_s z$$

We will use this equation to transform the horizontal derivatives in the momentum equation from z-coordinates to p-coordinates.

Horizontal momentum equation scaled for midlatitude large-scale motions.

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \frac{1}{\rho}\nabla p$$

Rate of change of velocity following the fluid motion.

Coriolis acceleration

Pressure gradient force (per unit mass)

To transform to pressure coordinates, we need to transform the pressure gradient term:

$$\nabla_p p = \nabla_z p + \frac{\partial p}{\partial z}\nabla_p z$$

$$\frac{\partial p}{\partial z} = -\rho g$$

$$\nabla_z p = -\frac{\partial p}{\partial z}\nabla_p z$$

$$\nabla_z p = \rho g \nabla_p z$$

$$-\frac{1}{\rho}\nabla_z p = -g\nabla_p z = -\nabla_p \Phi$$

Geopotential gradient

$$\boxed{\nabla_z p = \rho g \nabla_p z = \rho \nabla_p \Phi}$$

$$\left(\frac{\partial p}{\partial t}\right)_z = \rho g\left(\frac{\partial z}{\partial t}\right)_p = \rho\left(\frac{\partial \Phi}{\partial t}\right)_p$$

$$\rho\left(\frac{\partial \Phi}{\partial t}\right)_{sp} = \frac{1}{\rho}\frac{\partial p_s}{\partial t} \quad (8)$$

$$(8),(9) \rightarrow (7)$$

$$\nabla_{sp}\Phi = \frac{1}{\rho_s}\nabla p_s \quad (9)$$

$$\frac{\partial p_s}{\partial t} = -\int_0^{p_s} \nabla_p . \vec{V}_H dp - \vec{V}_{sH} . \nabla p_s \quad (7)$$

$$\frac{1}{\rho_s}\frac{\partial p_s}{\partial t} = -\frac{1}{\rho_s}\int_0^{p_s} \nabla_p . \vec{V}_H dp - \frac{1}{\rho_s}\vec{V}_{sH} . \nabla p_s$$

$$\left(\frac{\partial \phi}{\partial t}\right)_{sp} = -\alpha_s \int_0^{p_s} \nabla_p . \vec{V}_H dp - \vec{V}_{sH} . \nabla_{sp} \phi \qquad (10)$$

$$\begin{cases} p_s = p_0 = 1000\ hpa \\ \phi_{sp} = \phi_0 \end{cases}$$

$$\frac{\partial \phi_0}{\partial t} = -\alpha_0 \int_0^{p_s} \nabla_p . \vec{V}_H dp - \vec{V}_0 . \nabla \phi_0 \qquad (11)$$

$$\phi_0(t) \rightarrow \phi_0(t+\delta t) \rightarrow \frac{\partial \phi}{\partial p} = -\alpha \rightarrow \phi$$

$$\frac{\partial \phi}{\partial p} = -\alpha \quad \rightarrow \int_{\phi_0}^{\phi(p)} d\phi = -\alpha \int_{p_0}^{p} dpà \quad \rightarrow \int_{\phi_0}^{\phi(p)} d\phi = \int_{p}^{p_0} \hat{R}\theta dp$$

$$\phi(p) = \phi_0 + \int_{p}^{p_0} \hat{R}\theta dp \quad (12)$$

$$\hat{R} = \frac{R}{p}\left(\frac{p}{p_0}\right)^k = \frac{\alpha}{\theta}$$

$$\frac{\partial \omega}{\partial p} = -\nabla_p . \vec{V}_H \qquad \omega(p) = -\int_{0}^{p} \nabla_p . \vec{V}_H \; dp$$