
فیزیک 1
درس 24
صحرایی
گروه فیزیک دانشگاه رازی
http://www.razi.ac.ir/sahraei

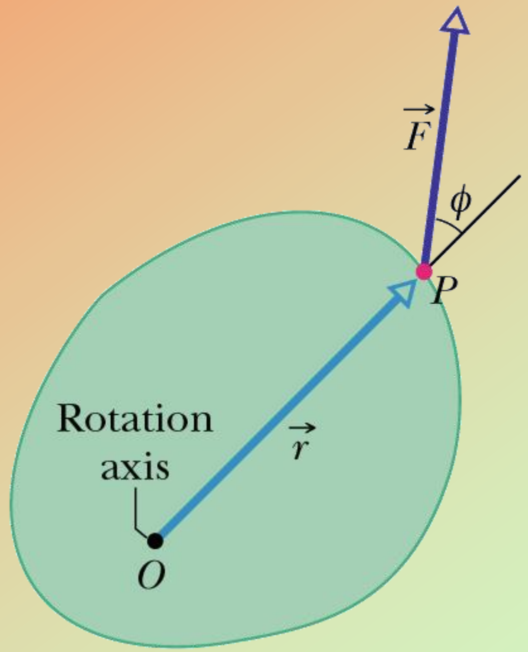
Torque

- **When you try to swing a door, it's clear that it is easier to move the door if you apply the force farther away from the axis of rotation (e.g., the hinges).**

- **It's also clear that the application of the force is most efficient if the force is applied perpendicular to the plane of the door.**

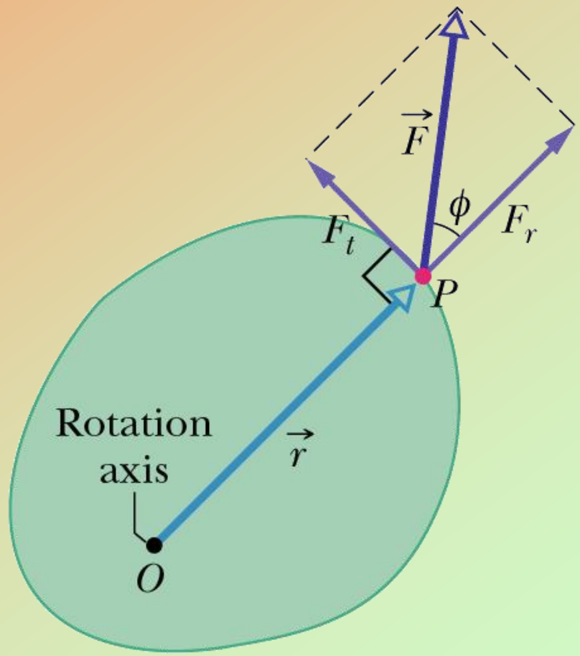
Torque

- Lets look at an object that is rotating about a fixed axis *O*
- A force *F* is applied at point *P* which is at a position *r* relative to the axis *O*
- Note also that the force *F* is applied at an angle *Φ* relative to the vector *r*
- For simplicity we also assume that the force *F* is in the plane of the screen



Torque

- Lets decompose the force into its components relative to the vector *r* :
  - The radial component is labeled F_r ($F\cos\Phi$)
  - The tangential component is labeled F_t ($F\sin\Phi$)



Torque

- We therefore define *torque* to be the product of these two values:

$$\tau = (r)(F \sin \phi)$$

- Torque comes from the Latin word meaning "to twist"

Torque

- By rearranging things a bit in the previous equation we can equivalently see that:

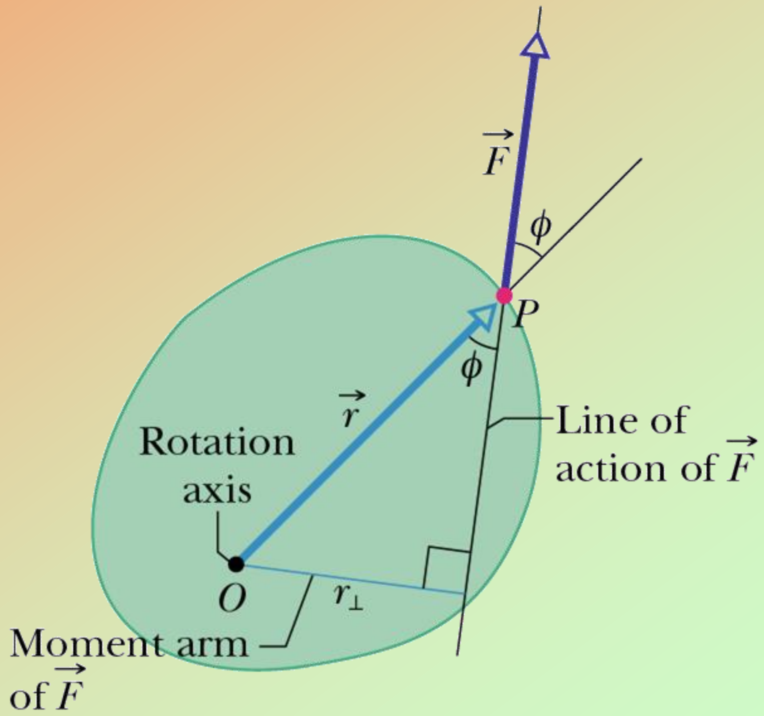
$$\tau = (r)(F \sin \phi) = rF_t$$

$$\tau = (r \sin \phi)(F) = r_\perp F$$

where $r_\perp$ is the perpendicular distance from the rotation axis O to an extended line running through the vector F at point P

Torque

- The extended line is called the *line of action* and the value $r_{\perp}$ is called the *moment arm*.
- Clearly if the force is applied completely tangentially, then the moment arm is just *r*.



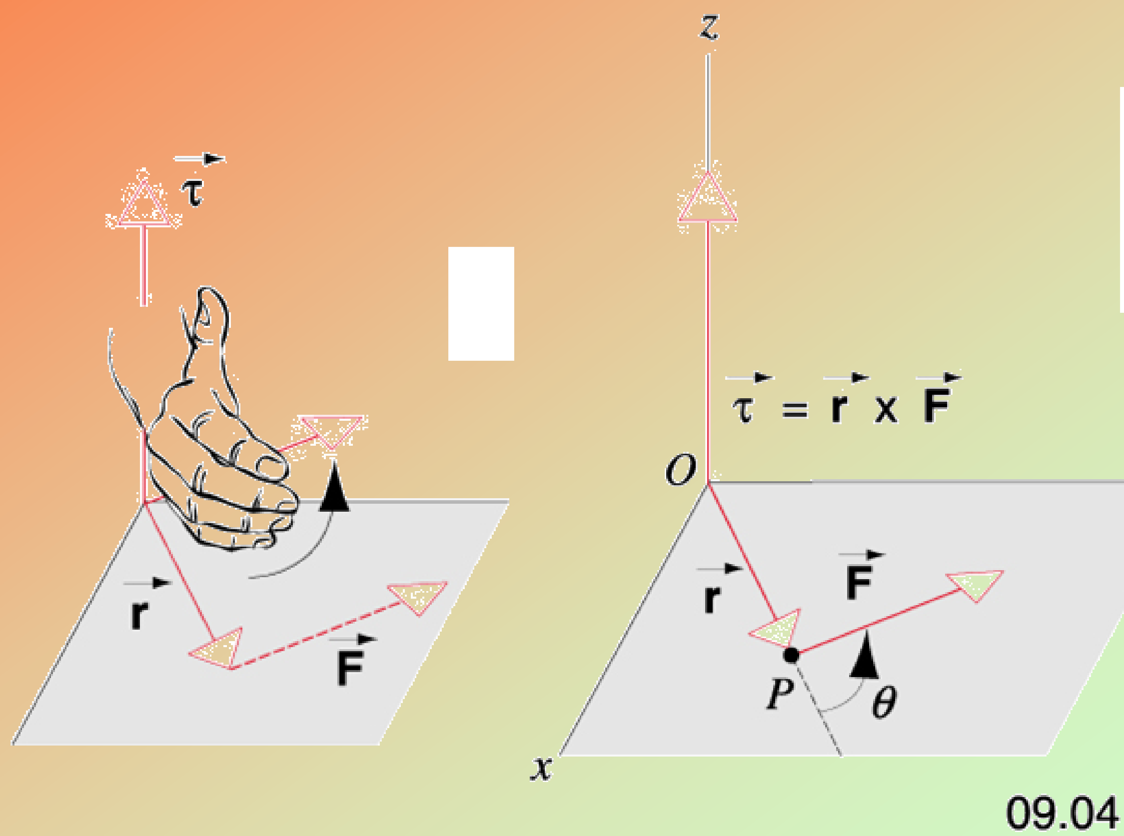
If an object rotates counterclockwise then the torque is positive – and vice versa (remember that clocks are negative…)

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\tau = rF\sin\theta$$

$$N.m \;,\; lb.ft$$

$$ML^2T^{-2}$$



Work can also be expressed in joules (1 J = 1 N·m), but torque is never expressed that way.

مثال: آونگی تشکیل شده است از جسمی به جرم $m=0.17\ kg$ که به انتهای میله صلبی به طول $L=1.25\ m$ و جرم ناچیز متصل است. مقدار گشتاور ناشی از گرانش حول نقطه O در لحظه ای که آونگ به اندازه $\theta=10^0$ از امتداد قائم منحرف شده چقدر است؟

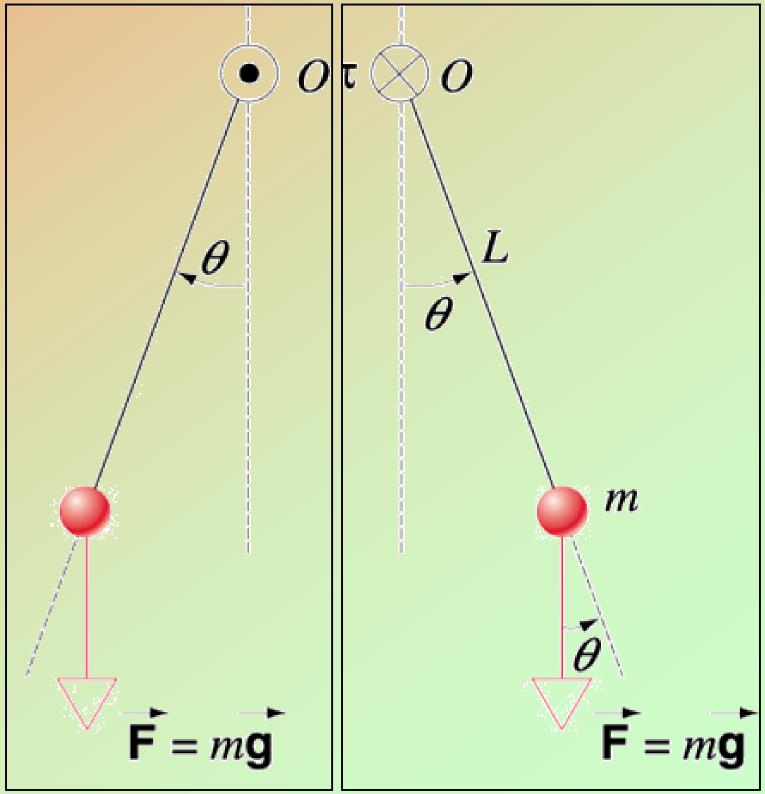
(ب) جهت گشتاور حول نقطه O در آن لحظه کدام است؟ آیا جهت گشتاور بستگی دارد به اینکه آونگ به کدام طرف خط قائم، چپ یا راست، جابجا می شود؟

$$\vec{\tau} = \vec{r} \times \vec{F} = rF\sin\theta$$

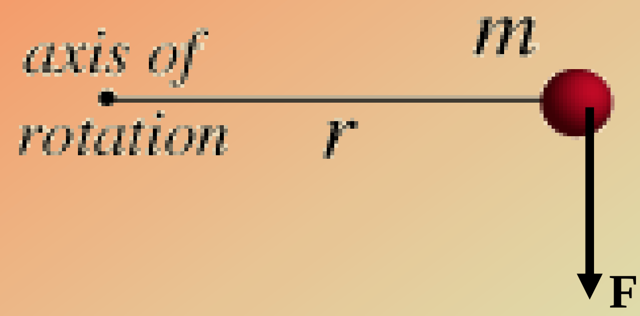
$$\tau = Lmg\sin\theta$$

$$= (1.25m)(0.17kg)(9.8m/s^2)$$

$$(\sin 10^0) = 0.36N.m$$



$$I = mr^2$$



$$\vec{\tau} = \vec{r} \times \vec{F} = rF\sin\theta$$

$$\tau = mar$$

$$\tau = I\alpha$$

$$mar = I\alpha$$

$$a_T = r\alpha$$

$$I = mr^2$$

Work and Rotational Kinetic Energy

Suppose that the change in the kinetic energy is the <u>only</u> change in the overall energy of the system – thus:

$$\Delta K = K_f - K_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = W$$

For motion confined to a single axis (let's say the *x* axis) we have:

$$W = \int_{x_i}^{x_f} F dx \qquad W = F\left(x_f - x_i\right)$$

And finally, the power is: $$P = \frac{dW}{dt} = Fv$$

In this case suppose that the change in the rotational kinetic energy is the only change in the overall energy of the system – thus:

$$\Delta K = K_f - K_i = \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_i^2 = W$$

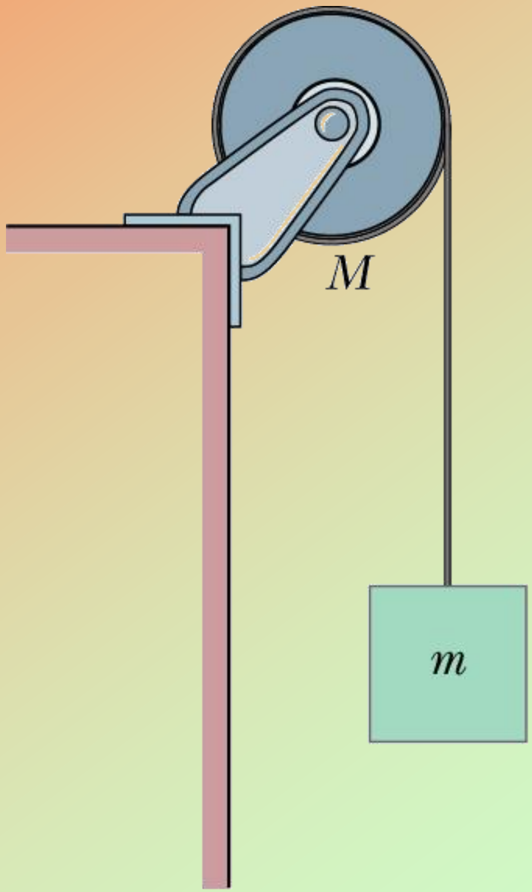
The corresponding equation for work in the rotational case is:

$$W = \int_{\theta_i}^{\theta_f} \tau d\theta \qquad W = \tau(\theta_f - \theta_i)$$

when the torque is constant and the angular displacement goes from θ_i to θ_f

$$P = \frac{dW}{dt} = \tau\omega$$

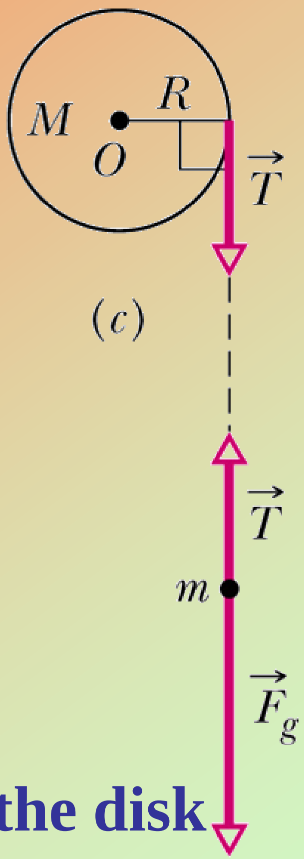
- We have a disk with mass $M = 2.5$ kg and a radius $R = 20$ cm mounted on a fixed, frictionless axle
- A block of mass $m = 1.2$ kg hangs from a massless string which is wrapped around the disk several times
- Find the acceleration of the block, the angular acceleration of the disk and the string tension



- **As usual, we begin by drawing the free body diagrams**
- **Starting with the block we see that our force equation turns out to be:**

$$T - mg = ma$$

- **We have the acceleration *a* in this equation, but we can't solve for it yet as we don't know the value of *T***



- **Moving on to the disk, we can see that the torque on the disk is:**

$$\tau = -RT$$

- **(since the disk is turning clockwise, the torque is negative)**

$$\tau_{net} = I\alpha \qquad I = \frac{1}{2}MR^2$$

$$-RT = \frac{1}{2}MR^2\alpha \qquad a_T = \alpha R \qquad \alpha = \frac{a}{R}$$

$$T = -\frac{1}{2}Ma \qquad T - mg = ma$$

$$a = -g\frac{2m}{M+2m} = -4.8\ \mathrm{m/s}^2$$

$$T = -\frac{1}{2}Ma = 6.0\ \mathrm{N}$$

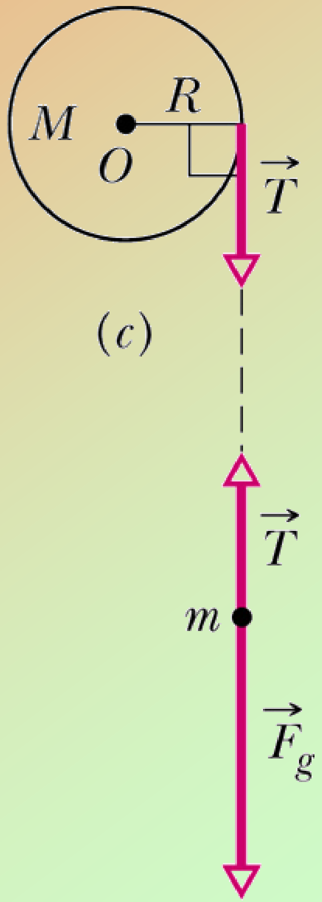
$$\alpha = \frac{a}{R} = \frac{-4.8 m/s^2}{0.2m} = -24 rad/s^2 = -3.8 rev/s^2$$

As a final check let's see what happens when *M* = 0:

$$a = -g\frac{2m}{M+2m} = -g$$

$$T = \frac{1}{2}Ma = 0$$

مثال فوق را از دیدگاه کار و انرژی بررسی کنید.



$$W_{net} = mgL$$

$$\Delta K = K_f - K_i = K_f = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$$

$$W_{net} = \Delta K$$

$$mgL = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 = \frac{1}{2}(\frac{1}{2}MR^2)(\frac{v}{R})^2 + \frac{1}{2}mv^2$$

$$v^2 = 2\left[\frac{2mg}{M+2m}\right]L \qquad v^2 = v_0{}^2 + 2ax$$

$$W_{net} = mgL - TL \qquad mgL - TL = \frac{1}{2}mv^2$$

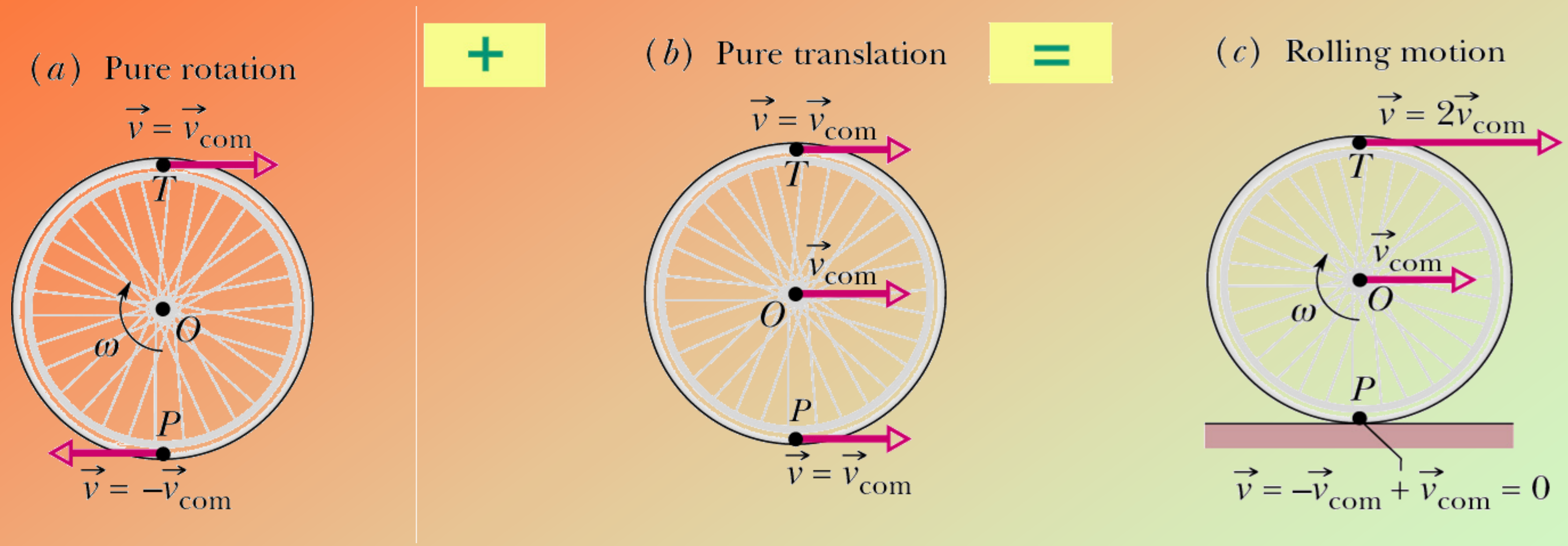
$$W_{net} = TR\phi = TL$$

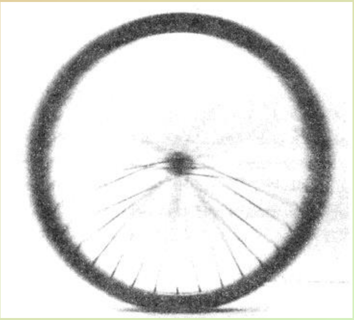
$$\Delta K = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(\frac{1}{2} MR^2\right)\frac{v^2}{R^2} = \frac{1}{4} Mv^2$$

$$TL = \frac{1}{4} Mv^2$$

Rolling

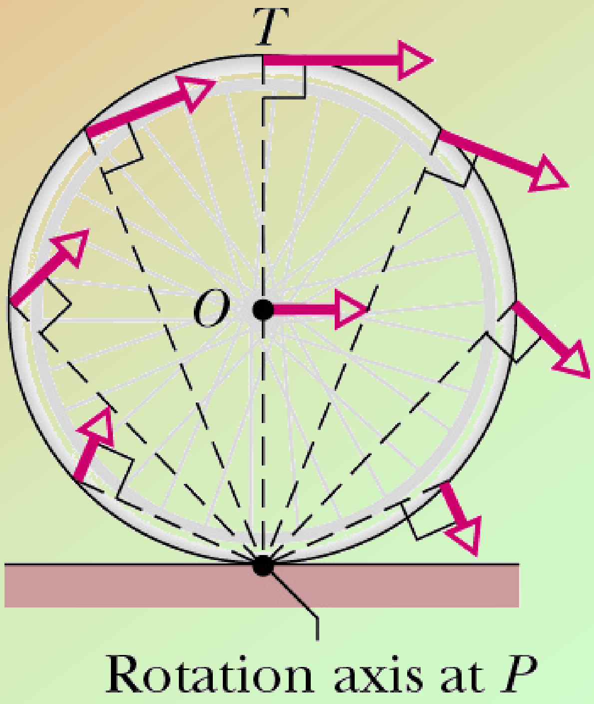
Here we can see that rolling motion is a combination of purely rotational and purely translational motions





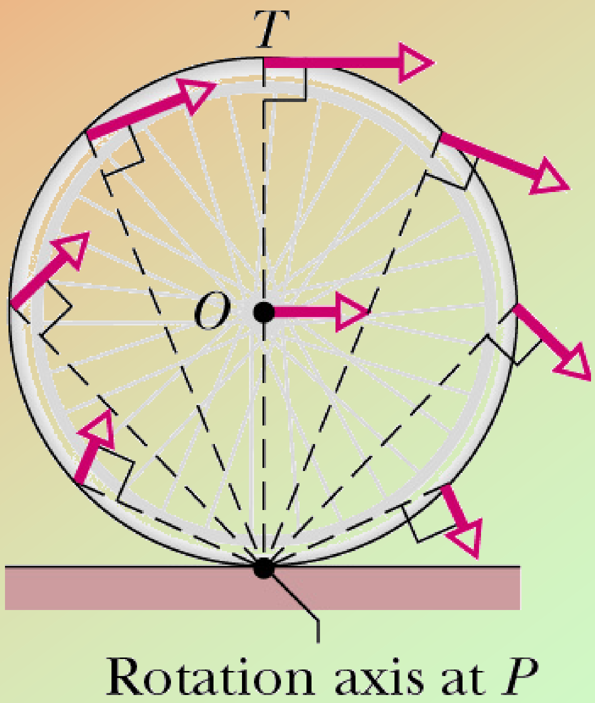
- There is another way to look at this however – a way which sees rolling as purely rotational
- In this case we view the rotation as being around the point where the wheel contacts the ground…

- The rotation axis is taken as the point *P*
- The vectors in the figure represent the instantaneous velocity of various points on the rolling wheel



The Kinetic Energy of Rolling

- We want to calculate the kinetic energy of a wheel rotating about the axis *P*
- We know that (since this is a purely rotational view of the problem) we have:



$$K = \frac{1}{2} I_P \omega^2 \qquad I_P = I_{cm} + MR^2$$

$$K = \frac{1}{2}(I_{cm} + MR^2)\omega^2$$

The Kinetic Energy of Rolling

- **Substituting for I_P in our equation for K and then expanding it out we get:**

$$K = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M R^2 \omega^2$$

- **and when we also substitute in $v_{cm} = \omega R$ we get:**

$$K = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M v_{cm}^2$$

The Kinetic Energy of Rolling

But this is exactly equivalent to the kinetic energy of a wheel rotating about it's CM added to the kinetic energy of a body's CM in translation:

$$K = \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} M v_{cm}^2$$

Rotational Kinetic Energy

Translational Kinetic Energy

A uniform solid cylindrical disk, of mass *M* = 1.4 kg and radius *R* = 8.5 cm, rolls smoothly across a horizontal table at a speed of 15 cm/s.

What is the kinetic energy *K* of the disk?

$$K = \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} M v_{cm}^2$$

speed of the center of mass So v_{cm} = 15 cm/s = 0.15 m/s

$$\omega = \frac{v_{cm}}{R} \qquad I_{cm} = \frac{1}{2} M R^2$$

$$K = \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} M v_{cm}^2$$

$$K = \frac{1}{2}\left(\frac{1}{2} M R^2\right)\left(\frac{v_{cm}}{R}\right)^2 + \frac{1}{2} M v_{cm}^2$$

$$K = \frac{3}{4} M v_{cm}^2$$

It doesn't depend on the radius of the disk

$$K = \frac{3}{4}(1.4\ \text{kg})(0.15\ \text{m/s})^2 = 0.024\ \text{J}$$

Rolling Motion

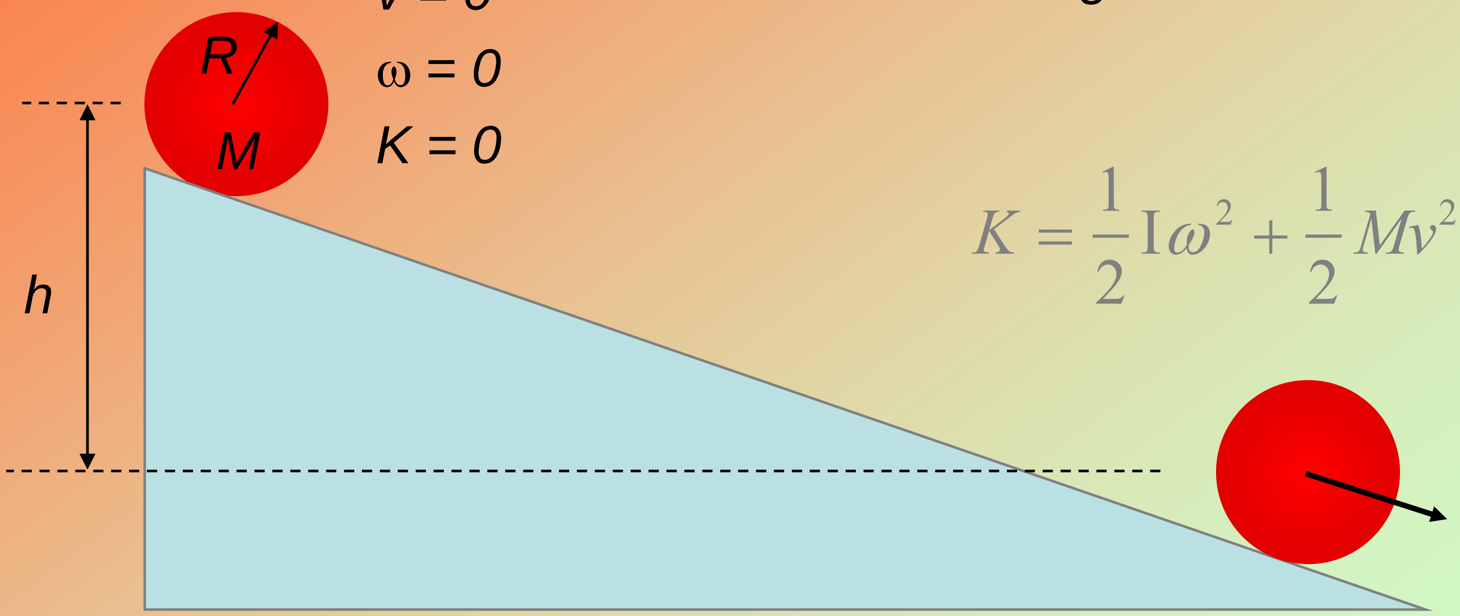
Objects of different *I* rolling down an inclined plane:

$v = 0$

$\omega = 0$

$K = 0$

$$\Delta K = -\Delta U = Mgh$$

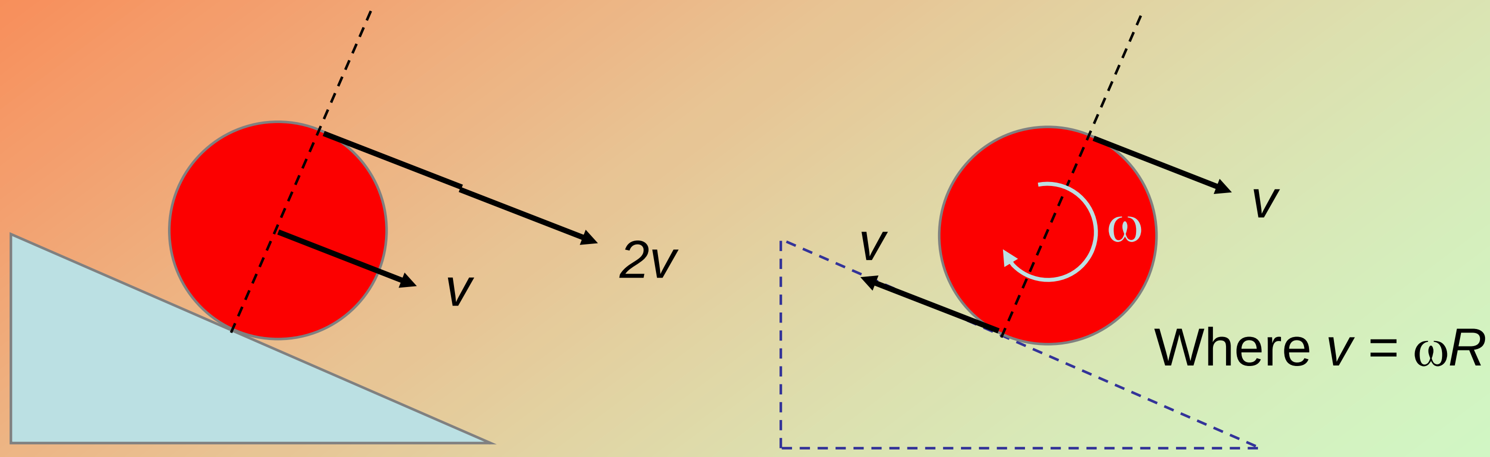


$$K = \frac{1}{2} I\omega^2 + \frac{1}{2} Mv^2$$

$v = \omega R$

Rolling...

- **If there is no slipping:**



In the lab reference frame

In the CM reference frame

Rolling...

$K = \frac{1}{2} I\omega^2 + \frac{1}{2} Mv^2$ Use $v = \omega R$ and $I = cMR^2$.

$$K = \frac{1}{2} cMR^2\omega^2 + \frac{1}{2} Mv^2 = \frac{1}{2}(c+1)Mv^2$$

hoop: c = 1

disk: c = 1/2

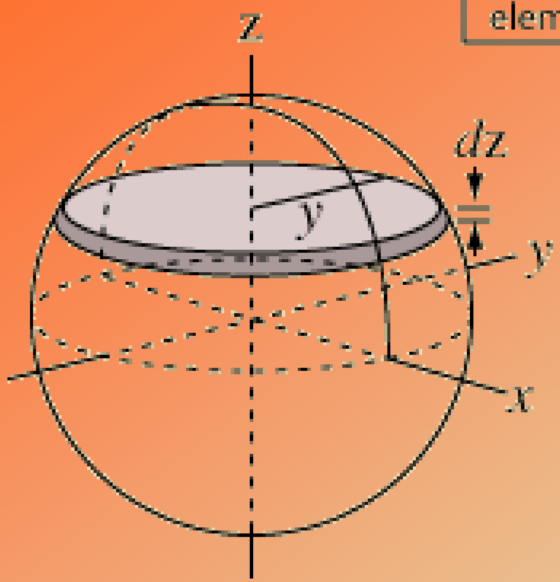
sphere: c = 2/5

etc...

So: $\frac{1}{2}(c+1)Mv^2 = Mgh$ ⇨ $v = \sqrt{2gh}\sqrt{\frac{1}{c+1}}$

The rolling speed is always lower than in the case of simple sliding since the kinetic energy is shared between CM motion and rotation.

Moment of Inertia: Sphere



Infinitesmal moment of inertia element

Moment of a thin disk of mass dm

Substituting area times height of disk for volume dV

$$dI = \frac{1}{2}y^2 dm = \frac{1}{2}y^2 \rho dV = \frac{1}{2}y^2 \rho\pi y^2 dz$$

Expressing mass dm in terms of density and volume.

Integrating the moment of inertia element from the bottom to the top of the sphere

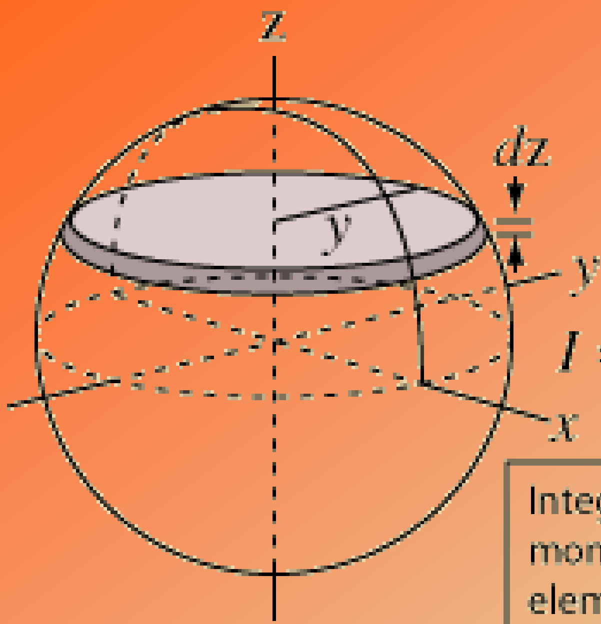
$$I = \frac{1}{2}\rho\pi \int_{-R}^{R} y^4 dz = \frac{1}{2}\rho\pi \int_{-R}^{R} (R^2 - z^2)^2 dz = \frac{8}{15}\rho\pi R^5$$

Radius = R

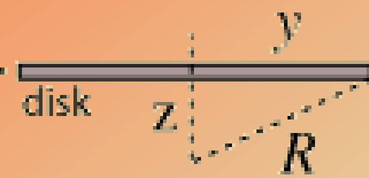
Mass = M

Density $= \rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3}$

$$I = \frac{8}{15}\left[\frac{M}{\frac{4}{3}\pi R^3}\right]\pi R^5 = \frac{2}{5}MR^2$$



Find y in terms of the variable of integration z



$$y^2 = R^2 - z^2$$

$$I = \frac{1}{2}\rho\pi \int_{-R}^{R} y^4 dz = \frac{1}{2}\rho\pi \int_{-R}^{R} (R^2 - z^2)^2 dz = \frac{8}{15}\rho\pi R^5$$

Integrating the moment of inertia element from the bottom to the top of the sphere

$$(R^2 - z^2)^2 = R^4 - 2R^2z^2 + z^4$$

Polynomial form integral

Radius = R

Mass = M

$$\int_{-R}^{R} \ldots\ dz = \left[R^4 z - \frac{2R^2z^3}{3} + \frac{z^5}{5} \right]_{-R}^{R} = 2R^5\left(\frac{15}{15} - \frac{10}{15} + \frac{3}{15}\right)$$

Density

$$= \rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3}$$

$$I = \frac{8}{15}\rho\pi R^5$$

- Rotational inertia involves not only the mass but also the distribution of mass for continuous masses
- Calculating the rotational inertia $I = \int r^2 dm$