# Dynamic Meteorology

## Lecture 16

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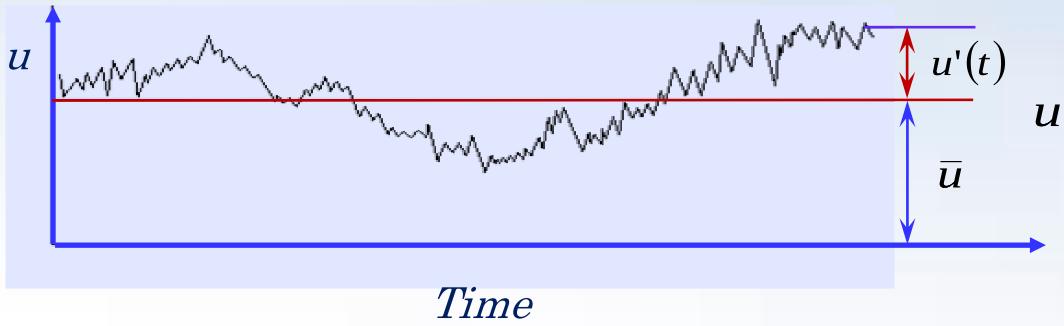
# Reynolds Averaging

In a turbulent fluid, a field variable such as velocity measured at a point generally fluctuates rapidly in time as eddies of various scales pass the point.

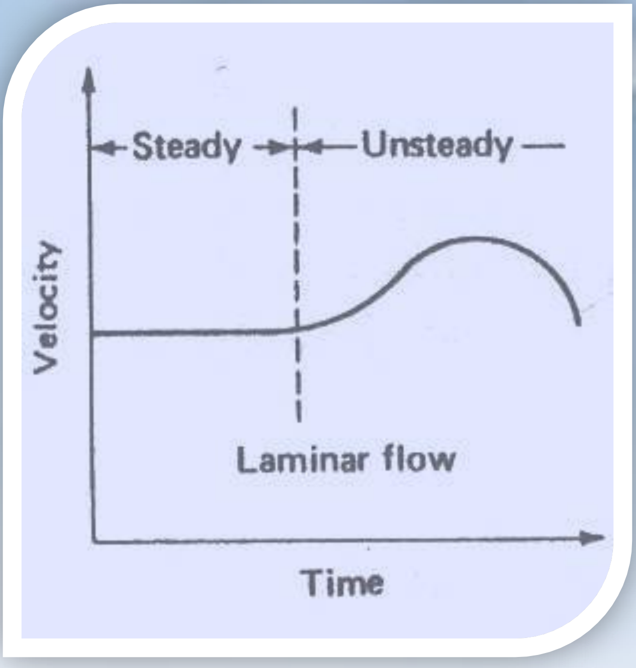
In order that measurements be truly representative of the large-scale flow, it is necessary to average the flow over an interval of time long enough to average out small-scale eddy fluctuations, but still short enough to preserve the trends in the large-scale flow field.

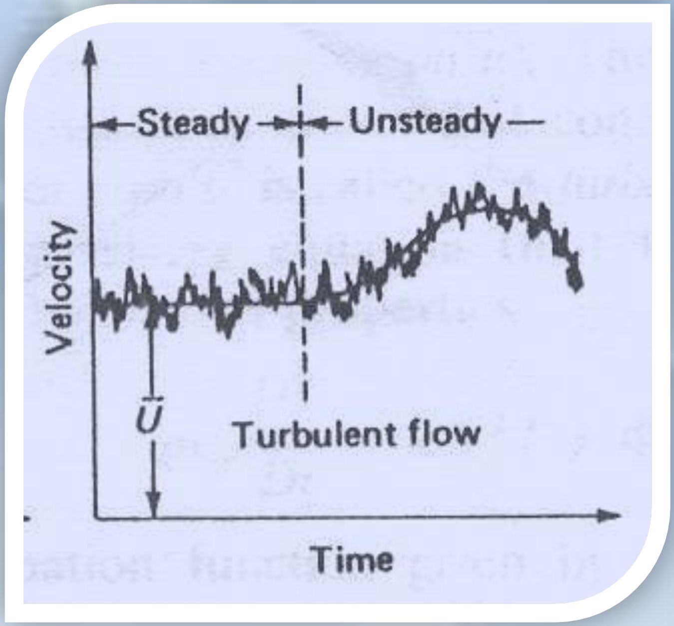
To do this we assume that the field variables can be separated into slowly varying mean fields and rapidly varying turbulent components.




$$u = \bar{u} + u'(t)$$

$\overline{u}$ itself may vary slowly with time as the following figure



$$\bar{u} = \frac{1}{T} \int_{t_o}^{t_o+T} u\,dt$$

Following the scheme introduced by Reynolds, we then assume that for any field variables, w and θ ,

for example, the corresponding means are indicated by overbars and the fluctuating components by primes.

$$w = \bar{w} + w' \qquad \theta = \bar{\theta} + \theta'$$

By definition, the means of the fluctuating components vanish; the product of a deviation with a mean also vanishes when the time average is applied. Thus,

$$\overline{w'\bar{\theta}} = \overline{w'}\bar{\theta} = 0$$

$$\overline{w\theta} = \overline{(\bar{w}+w')(\bar{\theta}+\theta')} = \bar{w}\bar{\theta} + \overline{w'\theta'}$$

These equations illustrate how the slowly varying velocities, $\bar{u}$ and $\bar{v}$

depend on the average of the product he average of the product

of the deviation components, i.e., $\overline{u'u'}, \overline{u'v'}$ ,etc. which are the

turbulent fluctuations or covariances, and represent eddy stresses.

If for example, on average the turbulent vertical velocity is upward (downward) where the potential temperature deviation is positive (negative), the product $\overline{w'\theta'}$ is positive and the variables are said to be positively correlated.

we rewrite the total derivative in each equation in flux form.
We combine the horizontal momentum equations

$$\rho \times \left[\frac{\partial u}{\partial t} + \vec{V} \cdot \nabla u - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}\right],$$

$$\rho \times \left[\frac{\partial v}{\partial t} + \vec{V} \cdot \nabla v + fv = -\frac{1}{\rho}\frac{\partial p}{\partial y}\right],$$

with the continuity equation

$$u \text{ or } v \times \left[\frac{\partial \rho}{\partial t} + \nabla \cdot \rho\vec{V} = 0\right],$$

to get the flux form of the momentum equations:

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho u\, v) + \frac{\partial}{\partial z}(\rho u\, w) - f\rho v = -\frac{\partial p}{\partial x}$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial}{\partial y}(\rho u\, v) + \frac{\partial}{\partial y}(\rho v^2) + \frac{\partial}{\partial z}(\rho v w) + f\rho u = -\frac{\partial p}{\partial y}$$

If we separate each dependent variable into mean and fluctuating parts

$$\vec{u} = \overline{u} + u', \qquad \vec{v} = \overline{v} + v',$$

$$\vec{w} = \overline{w} + w', \qquad \vec{p} = \overline{p} + p',$$

where it is assumed that $\rho' = 0$, so that $\rho = \bar{\rho}$, and then time average the flux forms of the momentum equation, we get

$$\frac{\partial}{\partial t}\overline{u} + \overline{u}\frac{\partial}{\partial x}\overline{u} + \overline{v}\frac{\partial}{\partial y}\overline{u} + \overline{w}\frac{\partial}{\partial z}\overline{u} - f\overline{v} = -\frac{1}{\rho}\frac{\partial}{\partial x}\overline{p} - \frac{1}{\rho}\Big[\underbrace{\frac{\partial}{\partial x}(\rho\overline{u'u'})}_{eddy\ stress} + \underbrace{\frac{\partial}{\partial y}(\rho\overline{u'v'})}_{eddy\ stress} + \underbrace{\frac{\partial}{\partial z}(\rho\overline{u'w'})}_{eddy\ stress}\Big]$$

$$\frac{\partial}{\partial t}\overline{v} + \overline{u}\frac{\partial}{\partial x}\overline{v} + \overline{v}\frac{\partial}{\partial y}\overline{v} + \overline{w}\frac{\partial}{\partial z}\overline{v} + f\overline{u} = -\frac{1}{\rho}\frac{\partial}{\partial y}\overline{p} - \frac{1}{\rho}\Big[\frac{\partial}{\partial x}(\rho\overline{u'v'}) + \frac{\partial}{\partial y}(\rho\overline{v'v'}) + \frac{\partial}{\partial z}(\rho\overline{v'w'})\Big]$$

where the eddy is a primed (′) quantity, the stress is a sheared quantity. If you have trouble understanding "flux", then look at the units of flux;

$$\rho u \rightarrow \frac{\mathrm{kg}}{\mathrm{l}^3}\frac{1}{\mathrm{s}} = \frac{\mathrm{kg}}{\mathrm{l}^2}\frac{1}{\mathrm{s}},$$

or the amount of mass flowing through a square area in some unit of time.

There are other ways of determining the flux form of the governing equations. For example, the term
on the left in (5.1) can be manipulated with the aid of the continuity equation (5.5) and the chain rule of differentiation to yield

$$\frac{Du}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + fv + F_{rx}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{du}{dt}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+u(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z})$$

$$=\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial uw}{\partial z}$$

Separating each dependent variable into mean and fluctuating parts, substituting into * and averaging then yields

$$\overline{\frac{du}{dt}}=\frac{\partial \bar{u}}{\partial t}+\frac{\partial}{\partial x}(\bar{u}\ \bar{u}+\overline{u'u'})+\frac{\partial}{\partial y}(\bar{u}\ \bar{v}+\overline{u'v'})+\frac{\partial}{\partial z}(\bar{u}\bar{w}+\overline{u'w'})$$

Noting that the mean velocity fields satisfy the continuity equation, we can rewrite

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\mathbf{0}$$

$$\overline{\frac{du}{dt}}=\frac{\bar{d}\bar{u}}{dt}+\frac{\partial}{\partial x}(\overline{u'u'})+\frac{\partial}{\partial y}(\overline{u'v'})+\frac{\partial}{\partial z}(\overline{u'w'})$$

$$\frac{\bar{d}}{dt}=\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}+\bar{v}\frac{\partial}{\partial y}+\bar{w}\frac{\partial}{\partial z}$$

is the rate of change following the mean motion. The mean equations thus have the form

$$\frac{\bar{d}\bar{u}}{dt}=-\frac{1}{\rho_0}\frac{\partial\bar{p}}{\partial x}+f\bar{v}-\left[\frac{\partial\overline{u'u'}}{\partial x}+\frac{\partial\overline{u'v'}}{\partial y}+\frac{\partial\overline{u'w'}}{\partial z}\right]+\bar{F}_{rx}$$

$$\frac{\bar{d}\bar{v}}{dt}=-\frac{1}{\rho_0}\frac{\partial\bar{p}}{\partial y}+f\bar{u}-\left[\frac{\partial\overline{u'v'}}{\partial x}+\frac{\partial\overline{v'v'}}{\partial y}+\frac{\partial\overline{v'w'}}{\partial z}\right]+\bar{F}_{ry}$$

$$\frac{\bar{d}\bar{w}}{dt}=-\frac{1}{\rho_0}\frac{\partial\bar{p}}{\partial z}+g\frac{\bar{\theta}}{\theta_0}-\left[\frac{\partial\overline{u'w'}}{\partial x}+\frac{\partial\overline{v'w'}}{\partial y}+\frac{\partial\overline{w'w'}}{\partial z}\right]+\bar{F}_{rz}$$

$$\frac{\bar{d}\bar{\theta}}{dt}=-\bar{w}\frac{d\theta_0}{dz}-\left[\frac{\partial\overline{u'\theta'}}{\partial x}+\frac{\partial\overline{v'\theta'}}{\partial y}+\frac{\partial\overline{w'\theta'}}{\partial z}\right]$$

$$\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}+\frac{\partial\bar{w}}{\partial z}=0$$

5 unknown mean variables, $\bar{u},\bar{v},\bar{w},\bar{\theta},\bar{p}$, there are unknown turbulent fluxes

The various covariance terms in square brackets in equations represent turbulent fluxes.

For example: $\overline{w'\theta'}$ is a vertical turbulent heat flux in kinematic form.

Similarly $\overline{w'u'} = \overline{u'w'}$ is a vertical turbulent flux of zonal momentum.

For many boundary layers the magnitudes of the turbulent flux divergence terms are of the same order as the other terms in equartions.

In such cases, it is not possible to neglect the turbulent flux terms even when only the mean flow is of direct interest.

Outside the boundary layer the turbulent fluxes are often sufficiently weak so that the terms in square brackets in equations can be neglected in the analysis of large-scale flows.

# PLANETARY BOUNDARY LAYER MOMENTUM EQUATIONS

For the special case of horizontally homogeneous turbulence above the viscous sublayer, molecular viscosity and horizontal turbulent momentum flux divergence terms can be neglected. The mean flow horizontal momentum equations

$$\frac{\bar{d}\bar{u}}{dt} = -\frac{1}{\rho_0}\frac{\partial\bar{p}}{\partial x} + f\bar{v} - \left[\frac{\partial\overline{u'u'}}{\partial x} + \frac{\partial\overline{u'v'}}{\partial y} + \frac{\partial\overline{u'w'}}{\partial z}\right] + \bar{F}_{rx}$$

$$\frac{\bar{d}\bar{v}}{dt} = -\frac{1}{\rho_0}\frac{\partial\bar{p}}{\partial y} + f\bar{u} - \left[\frac{\partial\overline{u'v'}}{\partial x} + \frac{\partial\overline{v'v'}}{\partial y} + \frac{\partial\overline{v'w'}}{\partial z}\right] + \bar{F}_{ry}$$

then become

$$\frac{\bar{d}\bar{u}}{dt} = -\frac{1}{\rho_0}\frac{\partial\bar{p}}{\partial x} + f\bar{v} - \frac{\partial\overline{u'w'}}{\partial z} \qquad \frac{\bar{d}\bar{v}}{dt} = -\frac{1}{\rho_0}\frac{\partial\bar{p}}{\partial y} + f\bar{u} - \frac{\partial\overline{v'w'}}{\partial z}$$

In general, these equ. can only be solved for $\bar{u}$ *and* $\bar{v}$ *if the vertical distribution* of the turbulent momentum flux is known.

Because this depends on the structure of the turbulence, no general solution is possible. Rather, a number of approximate semiempirical methods are used.

For midlatitude synoptic-scale motions, showed that to a first approximation the inertial acceleration terms, the terms on the left in equations can be neglected compared to the Coriolis force and pressure gradient force terms.

Outside the boundary layer, the resulting approximation is then simply geostrophic balance.

In the boundary layer the inertial terms are still small compared to the Coriolis force and pressure gradient force terms, but the turbulent flux terms must be included.

Thus, to a first approximation, planetary boundary layer equations express a three-way balance among the Coriolis force, the pressure gradient force, and the turbulent momentum flux divergence:

$$f(\bar{v} - \bar{v}_g) - \frac{\partial \overline{u'w'}}{\partial z} = 0 \qquad -f(\bar{u} - \bar{u}_g) - \frac{\partial \overline{v'w'}}{\partial z} = 0$$

$$\vec{V}_g \equiv \hat{k} \times \frac{1}{\rho f} \nabla p$$

Thus, knowledge of the pressure distribution at any time determines the geostrophic wind.

# Reynolds Stress

Stress: Force per unit area (e.g. N m$^{-2}$ or kg m$^{-1}$ s$^{-2}$)

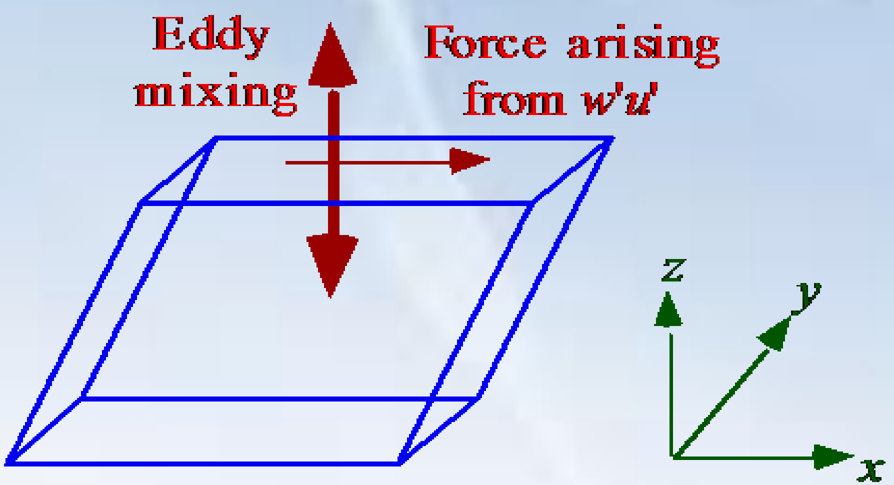
Reynolds stress: Stress that causes a parcel of air to deform during turbulent motion of air

Deformation by vertical momentum flux $\overline{w'u'}$

Stress from vertical transfer of turbulent *u*-momentum

$$\tau_{zx} = -\rho\overline{w'u'}$$

zx = stress acting in x-direction, along a plane (x-y) normal to the z-direction



# MOMENTUM FLUXES

Magnitude of Reynolds stress at ground surface

$$|\tau_z| = \rho\left[\left(\overline{w'u'}\right)^2 + \left(\overline{w'v'}\right)^2\right]^{1/2}$$

Kinematic vertical turbulent momentum flux ($m^2\ s^{-2}$)

$$\overline{w'u'} = -\frac{\tau_{zx}}{\rho} \qquad \overline{w'v'} = -\frac{\tau_{zy}}{\rho}$$

Friction wind speed ($m\ s^{-1}$)

Scaling param. for surface-layer vert. flux of horiz. momentum

$$u_* = \left[\left(\overline{w'u'}\right)^2 + \left(\overline{w'v'}\right)^2\right]^{1/4} = \left(|\tau_z|/\rho\right)^{1/2}$$


THANKS
FOR YOUR ATTENTION