Dynamic Meteorology 2

Lecture 9

Sahraei

Physics Department

Razi University

http://www.razi.ac.ir/sahraei

Stream Function

Incompressible fluid

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Definition of Stream Function $\quad u=-\frac{\partial \psi}{\partial y}\,; \quad v=\frac{\partial \psi}{\partial x}$

Substituting these in the irrotationality condition, we have

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0$$

$$\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}=0$$

Velocity Potential

Irrotational flow

$$\nabla \times \vec{V} = 0 \qquad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

Since the, $\nabla \times \vec{V} = 0$
the flow field is irrotational.

Definition of Velocity Potential

$$u = \frac{\partial \phi}{\partial x}; \quad v = \frac{\partial \phi}{\partial y}$$

Velocity potential is a powerful tool in analysing irrotational flows.

Continuity Equation

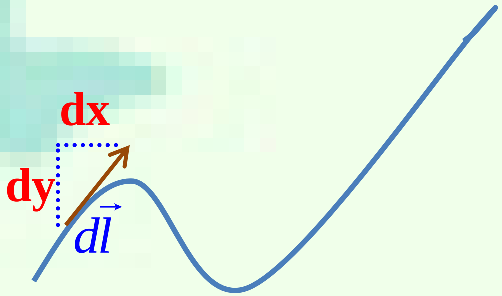
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \nabla^2 \phi = 0$$

As with stream functions we can have lines along which potential ϕ is constant. These are called Equipotential Lines of the flow.

Thus along a potential line $\phi = c$

Flow along a line $\vec{l}$

Consider a fluid particle moving along a line $\vec{l}$.



For each small displacement $d\vec{l}$

$$d\vec{l} = \hat{i}dx + \hat{j}dy$$

Where $\hat{i}$ and $\hat{j}$ are unit vectors in the *x* and *y* directions, respectively.

Since $d\vec{l}$ is parallel to $\vec{V}$, then the cross product must be zero.

$$\vec{V} = \hat{i}u + \hat{j}v$$

$$\vec{V} \times d\vec{l} = \left(\hat{i}u + \hat{j}v\right) \times \left(\hat{i}dx + \hat{j}dy\right) = \left(udy - vdx\right)\hat{k} = 0$$

$$\frac{dx}{u} = \frac{dy}{v}$$

Stream Function

Since $\dfrac{dx}{u} = \dfrac{dy}{v}$ must be satisfied along a line $\vec{l}$, such a line is called a streamline or flow line.

A mathematical construct called a stream function can describe flow associated with these lines.

The Stream Function $\psi(x, y)$ is defined as the function which is constant along a streamline, much as a potential function is constant along an equipotential line.

Since $\psi(x, y)$ is constant along a flow line, then for any $d\vec{l}$,

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0 \qquad \text{along the streamline}$$

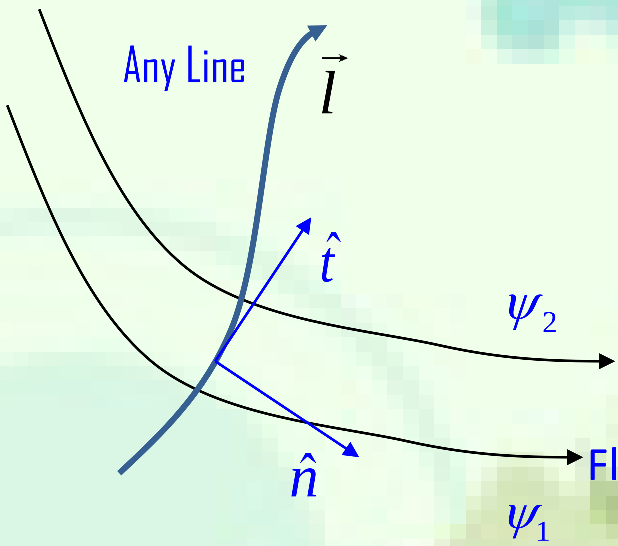
Stream Functions

$$\begin{cases} d\psi = \dfrac{\partial \psi}{\partial x} dx + \dfrac{\partial \psi}{\partial y} dy = 0 \\ \\ \dfrac{dx}{u} = \dfrac{dy}{v} \end{cases} \qquad \begin{cases} \dfrac{\partial \psi}{\partial x} dx = -\dfrac{\partial \psi}{\partial y} dy \\ \\ v dx = u dy \end{cases}$$

from which we can see that $$u = -\frac{\partial \psi}{\partial y} \quad , \quad v = \frac{\partial \psi}{\partial x}$$

So that if one can find the stream function, one can get the discharge by differentiation.

Interpreting Stream Functions



$\vec{l}$ is an arbitrary line

$\hat{t}$ is tangent to $\vec{l}$

$\hat{n}$ is normal to $\vec{l}$

$$\hat{t}.\hat{n} = 0$$

What is the flow that crosses $\vec{l}$ between flow Lines 1 and 2?

$$dQ = \vec{V}.\hat{n}dl = \left(u\hat{i} + v\hat{j}\right).\left(\frac{dy}{dl}\hat{i} - \frac{dx}{dl}\hat{j}\right)dl$$

for each increment $d\vec{l}$:

$$= udy - vdx = -\frac{\partial \psi}{\partial y}dy - \frac{\partial \psi}{\partial x}dx = -d\psi$$

$$\hat{t} = \frac{dx}{dl}\hat{i} + \frac{dy}{dl}\hat{j}$$

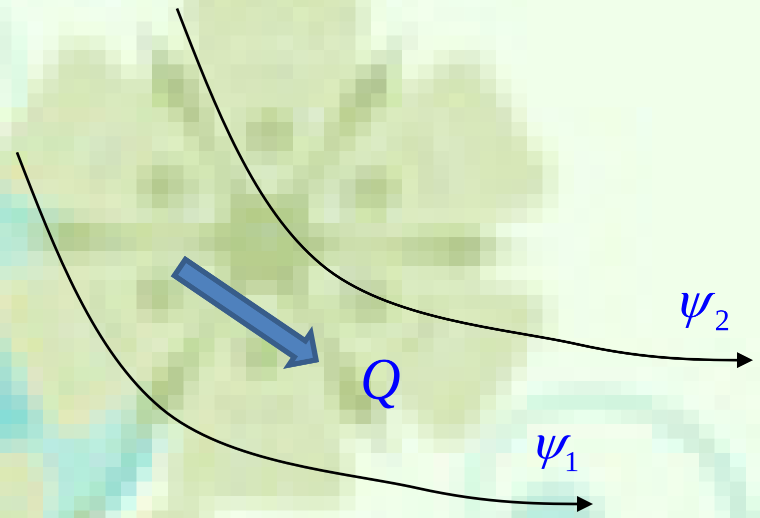
Discharge Between Lines 1, 2

$$\hat{n} = \frac{dy}{dl}\hat{i} - \frac{dx}{dl}\hat{j}$$

If we integrate along line $\vec{l}$, between flow lines 1 and 2 we will get the total flow across the line.

$$Q=\int_{\psi_1}^{\psi_2} dQ = \int_{\psi_1}^{\psi_2} -d\psi = -\psi\Big|_{\psi_1}^{\psi_2} = \int_{\psi_1}^{\psi_2} -d\psi = -\psi\Big|_{\psi_1}^{\psi_2} = \psi_1-\psi_2$$

Properties of Stream Function



Flow field variables are found by ϕ and ψ

We notice that velocity potential ϕ and stream function ψ are connected with velocity components.

It is necessary to bring out the similarities and differences between them.

Differentiating potential function ϕ in the same direction as velocities

Differentiating stream function ψ in direction normal to velocities

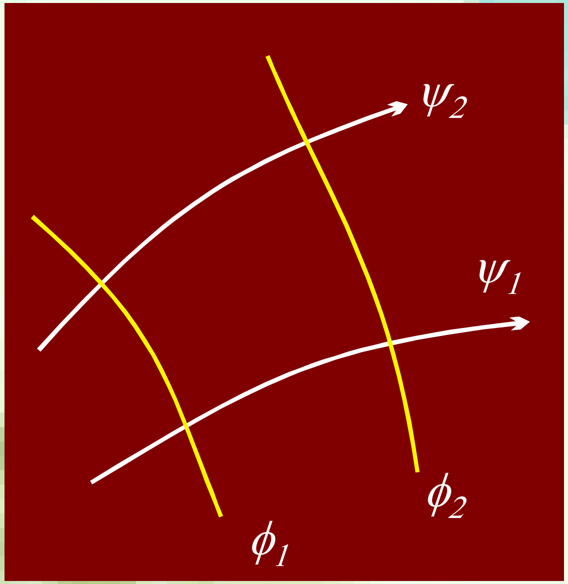
Potential function ϕ applies for irrotational flow only

Stream function ψ applies for rotational or irrotational flows

Potential function ϕ applies for 2D flows [$\phi(x,y)$ or $\phi(r,\theta)$] and 3D flows [$\phi(x,y,z)$ or $\phi(r,\theta, \phi)$]

Stream function ψ applies for 2D $\psi(x,y)$ or $\psi(r, \theta)$ flows only

Properties of Stream Function



Stream lines (ψ =constant) and equipotential lines (ϕ =constant) are always perpendicular to each other

همواره خطوط هم‌پتانسیل (هم‌فشار) بر خطوط جریان عمود می‌باشند. به عبارتی تابع جریان ϕ و تابع پتانسیل ψ متعامدند.

Slope of a line with ψ=constant is the negative reciprocal of the slope of a line with ϕ=constant

Stream Function- Physical meaning

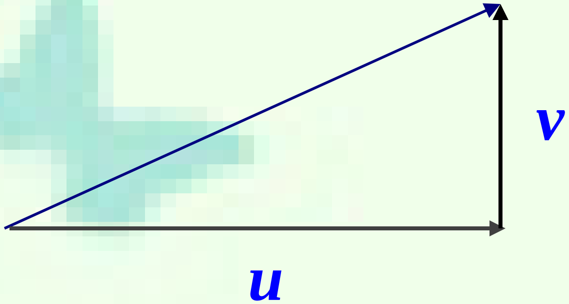
Statement: In 2D (viscous or inviscid) flow (incompressible flow OR steady state compressible flow), ψ = constant represents the streamline. (Properties of Stream Function)

Proof

If ψ = constant, then $d\psi = 0$

$$d\psi = \left(\frac{\partial \psi}{\partial x}\right)dx + \left(\frac{\partial \psi}{\partial y}\right)dy \quad = (-v)dx + (u)dy = 0$$

If ψ = constant, then $$\left(\frac{dy}{dx}\right)_{\psi} = \frac{v}{u}$$



Along a streamline

Along an equipotential line

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy \quad = udx + vdy = 0$$

$$\left(\frac{dy}{dx}\right)_{\phi} = -\frac{u}{v} \qquad \left(\frac{dy}{dx}\right)_{\phi}\left(\frac{dy}{dx}\right)_{\psi} = -1$$

showing that equipotential lines and streamlines are orthogonal to each other.

This enables one to calculate the stream function when the velocity potential is given and vice versa.

Fig. shows the flow through a bend where the streamlines and the equipotential lines have been plotted.