Dynamic Meteorology 2

Lecture 8

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Isentropic Coordinate System

Dry adiabatic motion

$\Downarrow$

Conservation of potential temperature

$\Downarrow$

Conservation of entropy

$\Downarrow$

"isentropic"

Potential temperature <u>is conserved</u> during an adiabatic process.

$$\theta = T\left(\frac{1000}{P}\right)^{\frac{R_d}{C_p}}$$

An adiabatic process is isentropic, that is, a process in which entropy is conserved

Entropy = $C_p \ln(\theta)$ + constant = 0

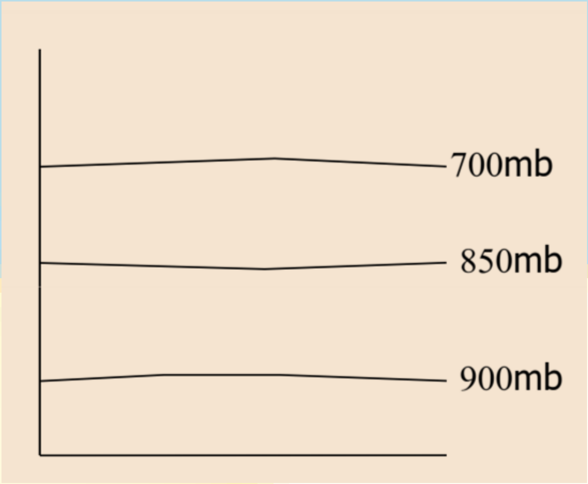
Potential temperature <u>is not conserved</u> when

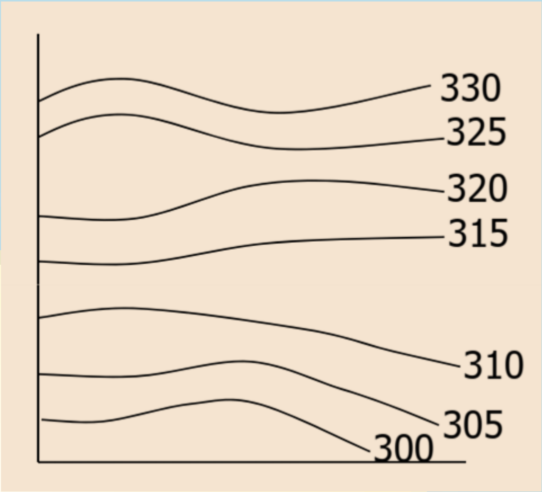
1) diabatic heating or cooling occurs or

2) mixing of air parcels with different properties occurs

Examples of diabatic processes: condensation, evaporation, sensible heating from surface, radiative heating, radiative cooling

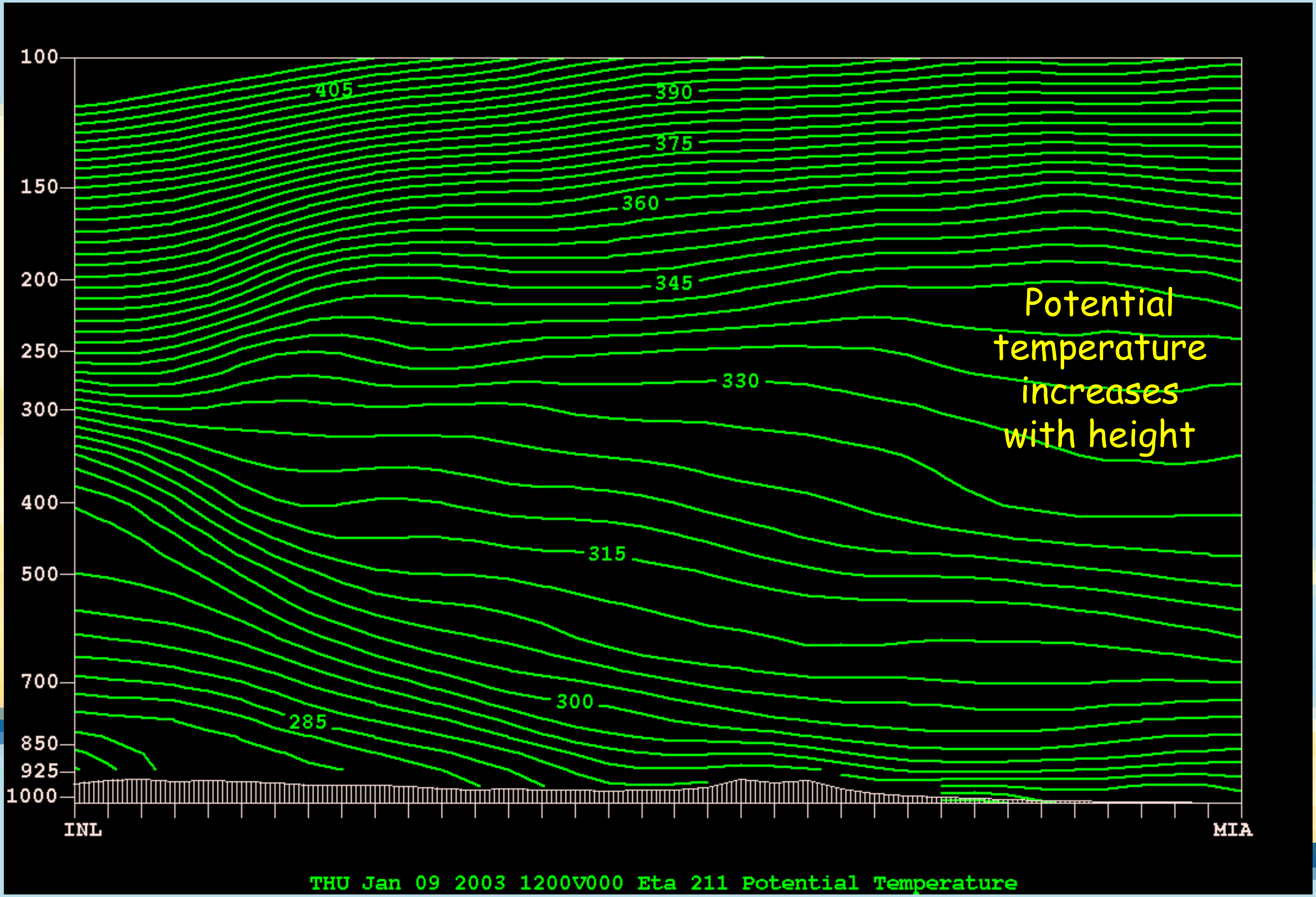
Comparing pressure coords. to isentropic coords.



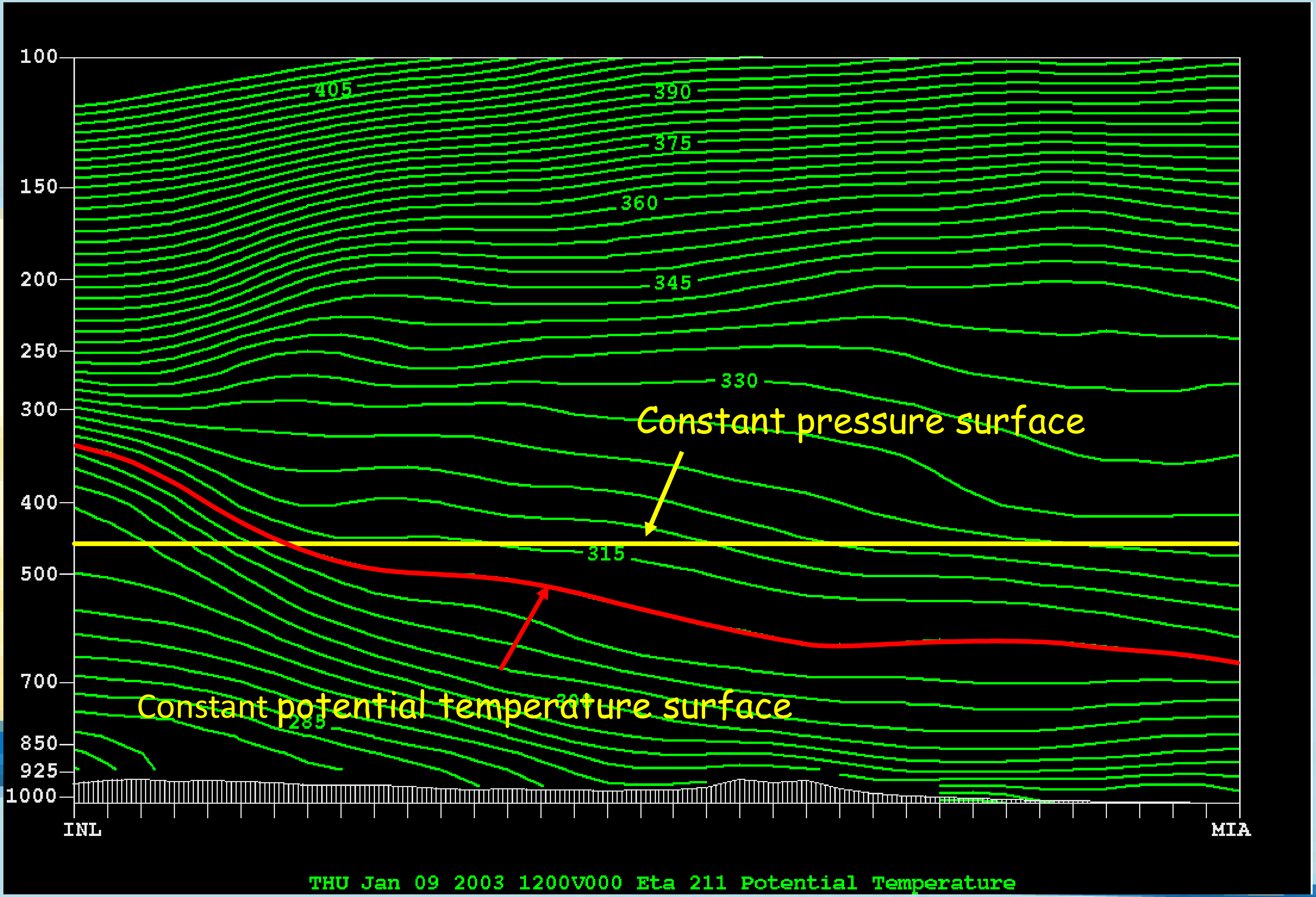


Potential temperature as a vertical coordinate

The troposphere, except in shallow, narrow, rare locations, is stable to dry processes. For the purpose of synoptic analysis, these areas can be ignored and potential temperature used as a vertical coordinate.



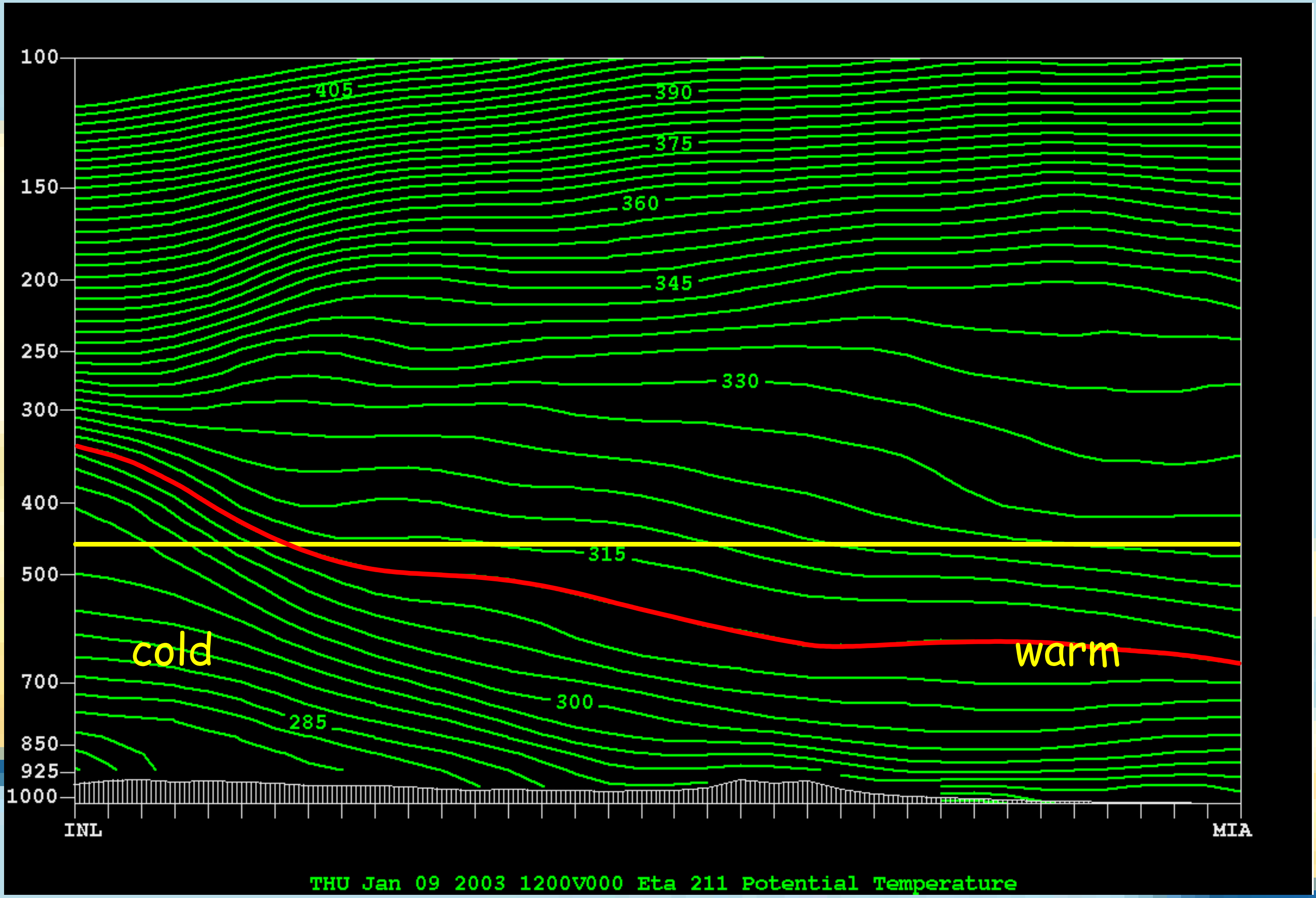
Isentropic Analyses are done on constant θ surfaces, rather than constant P or z



Note that:

1) isentropic surfaces slope downward toward warm air (opposite the slope of pressure surfaces)

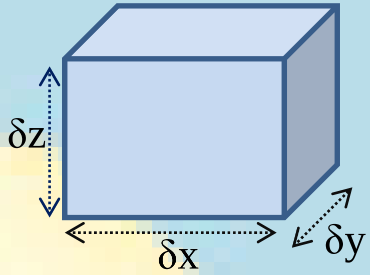
2) Isentropes slope much more steeply than pressure surfaces given the same thermal gradient.



The continuity equation in isentropic coordinates

δx, δy, $\delta\theta$ Isentropic coordinates

$$\delta m = \rho\ \delta v = \rho\ \delta A\ \delta z$$



$$\rho \delta z = -\frac{1}{g}\delta p$$

$$\rho\ \delta A\ \delta z = (-\frac{1}{g}\delta p)\delta A$$

$$\delta p = \frac{\partial p}{\partial \theta}\delta\theta$$ Taylor series

$$\rho\ \delta A\ \delta z = (-\frac{1}{g}\frac{\partial p}{\partial \theta})\delta A\ \delta\theta = \sigma\ \delta A\ \delta\theta$$

Here the density in (x, y, θ) space is defined as $\sigma = -\frac{1}{g}\frac{\partial p}{\partial \theta}$

The Eulerian form (fixed coordinate system) of the continuity equation

$$-\frac{1}{g}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \theta}\right) = -\nabla_{\theta}\cdot\left(\frac{\partial p}{\partial \theta}\vec{V}\right)$$

The Lagrangian form (moving coordinate system) of the continuity equation,

$$\frac{d}{dt}\left(\frac{\partial p}{\partial \theta}\right) = -\frac{\partial p}{\partial \theta}\nabla_{\theta}\cdot\vec{V}$$

$$\frac{d}{dt}\left(\frac{\partial p}{\partial \theta}\right) = \frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \theta}\right) + \vec{V}\cdot\nabla_{\theta}\left(\frac{\partial p}{\partial \theta}\right)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \theta}\right) = -\nabla_{\theta}\cdot\left(\frac{\partial p}{\partial \theta}\vec{V}\right)$$

$$\frac{\partial \theta}{\partial t} + \vec{V}\cdot\nabla\theta = 0 \qquad \frac{\partial \theta_s}{\partial t} + \vec{V}_s\cdot\nabla\theta_s = 0$$

ERTEL POTENTIAL VORTICITY IN ISENTROPIC COORDINATES

We consider here a more detailed treatment of the Ertel potential vorticity in isentropic coordinates, including nonconservative effects due to sources of momentum and entropy.

We begin with a description of the basic conservation laws in isentropic coordinates.

Equations of Motion in Isentropic Coordinates

If the atmosphere is stably stratified so that potential temperature Θ is a monotonically increasing function of height, Θ may be used as an independent vertical coordinate.

The Horizontal Momentum Equation

$$\frac{d\vec{V}}{dt} + f\hat{k} \times \vec{V} = -\nabla_p \Phi$$

where ∇_p is the horizontal gradient operator applied with pressure held constant.

The Vorticity Equation in Isobaric Coordinates

The horizontal momentum equation in isentropic coordinates may be obtained by transforming the isobaric form

$$(\vec{V}.\nabla)\vec{V} = \nabla(\frac{\vec{V}.\vec{V}}{2}) + \zeta\hat{k}\times\vec{V}$$

$$\zeta \equiv \hat{k}.(\nabla\times\vec{V})$$

$$\frac{\partial\vec{V}}{\partial t} \equiv -\nabla(\frac{\vec{V}.\vec{V}}{2} + \Phi) - (\zeta + f)\hat{k}\times\vec{V} - \omega\frac{\partial\vec{V}}{\partial p}$$

to yield $$\frac{\partial \vec{V}}{\partial t} + \nabla_\theta\left(\frac{\vec{V}.\vec{V}}{2} + \psi\right) + (\varsigma_\theta + f)\hat{k} \times \vec{V} = -\dot{\theta}\frac{\partial \vec{V}}{\partial \theta} + \vec{F}_r$$

Where ∇_θ is the gradient on in isentropic surface

$\varsigma_\theta \equiv \hat{k}.\nabla_\theta \times \vec{V}$ is the isentropic relative vorticity

and $\psi \equiv c_p T + \Phi$ is the *Montgomery streamfunction*

We have included a frictional term F_r on the right side, along with the diabatic vertical advection term.

The comtinuity eqation can be derived with the aid of

$$\frac{\partial \sigma}{\partial t} + \nabla_\theta.(\sigma\vec{V}) = -\frac{\partial}{\partial \theta}(\sigma\dot{\theta})$$

The ψ and σ fields are linked through the pressure field by the hydrostatic equation, which in the isentropic system takes the form

$$\frac{\partial \psi}{\partial \theta} = \Pi(p) \equiv c_p \left(\frac{p}{p_s}\right)^{R/c_p} = c_p \frac{T}{\theta} \qquad *$$

Where Π is called the *Exner function*.

Equations $$\sigma \equiv -g^{-1} \frac{\partial p}{\partial \theta}$$

through * form a closed set for prediction of $\vec{V},\ \sigma,\ \psi$ and ***p***.

provided that $\dot{\theta}$ and $\vec{F}_r$ are known.

The Potential Vorticity Equation

$$\frac{\partial \vec{V}}{\partial t} + \nabla_\theta \left(\frac{\vec{V}.\vec{V}}{2} + \psi\right) + (\varsigma_\theta + f)\hat{k} \times \vec{V} = -\dot{\theta}\frac{\partial \vec{V}}{\partial \theta} + \vec{F}_r \qquad *$$

If we take $\hat{k}.\nabla_\theta \times *$

and rearrange the resulting terms, we obtain this isentropic vorticity equation:

$$\frac{d}{dt}(\varsigma_\theta + f) + (\varsigma_\theta + f)\nabla_\theta.\vec{V} = \hat{k}.\nabla_\theta \times \left(\vec{F}_r - \dot{\theta}\frac{\partial \vec{V}}{\partial \theta}\right)$$

where $$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{V}.\nabla_\theta$$

is the total derivative following the horizontal motion on an isentropic surface.

Noting that $$\sigma^{-2}\frac{\partial \sigma}{\partial t} = -\frac{\partial}{\partial t}\sigma^{-1}$$

we can rewrite $$\frac{\partial \sigma}{\partial t} + \nabla_{\theta}.(\sigma \vec{V}) = -\frac{\partial}{\partial \theta}(\sigma \dot{\theta})$$

in the form $$\frac{d}{dt}(\sigma^{-1}) - (\sigma^{-1})\nabla_{\theta}.\vec{V} = \sigma^{-2}\frac{\partial}{\partial \theta}(\sigma \dot{\theta})$$

multiplying each term in $$\frac{d}{dt}(\varsigma_{\theta} + f) + (\varsigma_{\theta} + f)\nabla_{\theta}.\vec{V} = \hat{k}.\nabla_{\theta} \times (\vec{F}_r - \dot{\theta}\frac{\partial \vec{V}}{\partial \theta})$$

by σ^{-1}

and in $$\frac{d}{dt}(\sigma^{-1}) - (\sigma^{-1})\nabla_{\theta}.\vec{V} = \sigma^{-2}\frac{\partial}{\partial \theta}(\sigma \dot{\theta})$$

by $(\varsigma_{\theta} + f)$

and adding, we obtain the desired conservation law:

$$\frac{d\Pi}{dt} = \frac{\partial \Pi}{\partial t} + \vec{V}.\nabla_{\theta}\Pi = \frac{\Pi}{\sigma}\frac{\partial}{\partial \theta}(\sigma \dot{\theta}) + \sigma^{-1}\hat{k}.\nabla_{\theta} \times (\vec{F}_r - \dot{\theta}\frac{\partial \vec{V}}{\partial \theta})$$

$$\Pi \equiv \frac{(\zeta_\theta + f)}{\sigma}$$ is the Ertel potential vorticity

$$\frac{d\Pi}{dt} = \frac{\partial \Pi}{\partial t} + \vec{V}.\nabla_\theta \Pi = \frac{\Pi}{\sigma}\frac{\partial}{\partial \theta}(\sigma\dot{\theta}) + \sigma^{-1}\hat{k}.\nabla_\theta \times (\vec{F}_r - \dot{\theta}\frac{\partial \vec{V}}{\partial \theta})$$

If the diabatic and frictional terms on the right side of equation can be evaluated, it is possible to determine the evolution of Π following the *horizontal motion on an isentropic* surface.
When the diabatic and frictional terms are small, potential vorticity is approximately conserved following the motion on isentropic surfaces.

Weather disturbances that have sharp gradients in dynamical fields, such as jets and fronts, are associated with large anomalies in the Ertel potential vorticity.
In the upper troposphere such anomalies tend to be advected rapidly under nearly adiabatic conditions.
Thus, the potential vorticity anomaly patterns are conserved materially on isentropic surfaces. This material conservation property makes potential vorticity anomalies particularly useful in identifying and tracing the evolution of meteorological disturbances.

Integral Constraints on Isentropic Vorticity

The isentropic vorticity equation

$$\frac{d}{dt}(\varsigma_\theta + f) + (\varsigma_\theta + f)\nabla_\theta.\vec{V} = \hat{k}.\nabla_\theta \times (\vec{F}_r - \dot{\theta}\frac{\partial \vec{V}}{\partial \theta})$$

can be written in the form

$$\frac{\partial \varsigma_\theta}{\partial t} = -\nabla_\theta.\left[(\varsigma_\theta + f)\vec{V}\right] + \hat{k}.\nabla_\theta \times (\vec{F}_r - \dot{\theta}\frac{\partial \vec{V}}{\partial \theta}) \quad *$$

Using the fact that any vector A satisfies the relationship

$$\hat{k}.(\nabla_\theta \times \vec{A}) = \nabla_\theta.(\vec{A} \times \hat{k})$$

we can rewrite * in the form

$$\frac{\partial \varsigma_\theta}{\partial t} = -\nabla_\theta.\left[(\varsigma_\theta + f)\vec{V} - (\vec{F}_r - \dot{\theta}\frac{\partial \vec{V}}{\partial \theta}) \times \hat{k}\right]$$

This Equation expresses the remarkable fact that isentropic vorticity can only be changed by the divergence or convergence of the horizontal flux vector in brackets on the right side.

The vorticity cannot be changed by vertical transfer across the isentropes.

Furthermore, integration of this equation over the area of an isentropic Surface and application of the divergence theorem show that for an Isentropic that doesnot intersect the surface of Earth, the global average of ς_θ is constant.

Furthermore, integration of ς_θ over the sphere shows that the global average ς_θ is exactly zero.

Vorticity on such an isentrope is neither created nor destroyed, it is merely concentrated or diluted by horizontal fluxes along the isentropes.