Dynamic Meteorology 2
Lecture 2

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The Vorticity Equation

The previous section discussed kinematic properties of vorticity.

This section addresses vorticity dynamics using the equations of motion to determine contributions to the time rate of change of vorticity.

Cartesian Coordinate Form

For motions of synoptic scale, the vorticity equation can be derived using the approximate horizontal momentum equations.

We differentiate the zonal component equation with respect to y *and the meridional componentequation* with respect to x:

Vector momentum equation in rotating coordinates

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

Rate of change of relative velocity following the relative motion in a rotating reference frame

Coriolis acceleration

Pressure gradient force (per unit mass)

Gravity term (gravitation + centrifugal)

Friction

$$\frac{du}{dt} = \ldots$$ x-component momentum equation

$$\frac{dv}{dt} = \ldots$$ y-component momentum equation

Above the boundary layer, all horizontal parcel accelertions can be understood by comparing the magnitude and direction of the pressure gradient and coriolis forces.

$$\begin{cases} \dfrac{du}{dt} = -\dfrac{1}{\rho}\dfrac{\partial P}{\partial x} + fv \\ \dfrac{dv}{dt} = -\dfrac{1}{\rho}\dfrac{\partial P}{\partial y} - fu \end{cases}$$

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z} - fv = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z} + fu = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} \end{cases}$$

$$\frac{\partial}{\partial x}\text{[y-component momentum equation]}$$

$$-\frac{\partial}{\partial y}\text{[x-component momentum equation]} =$$

$$\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu=-\frac{1}{\rho}\frac{\partial p}{\partial y}\right]$$

$$-\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x}\right]$$

$$\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu=-\frac{1}{\rho}\frac{\partial p}{\partial y}\right]$$

$$\frac{\partial}{\partial x}\frac{\partial v}{\partial t}+u\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}+v\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial v}{\partial y}\frac{\partial v}{\partial x}+w\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial v}{\partial z}\frac{\partial w}{\partial x}+f\frac{\partial u}{\partial x}+u\frac{\partial f}{\partial x}=-\frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y}+\frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}\right)$$

$$-\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x}\right]$$

$$-\left[\frac{\partial}{\partial y}\frac{\partial u}{\partial t}+u\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+v\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+w\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial u}{\partial z}\frac{\partial w}{\partial y}-f\frac{\partial v}{\partial y}-v\frac{\partial f}{\partial y}=-\frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y}+\frac{1}{\rho^2}\left(\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)\right]$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+w\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

$$+\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+v\frac{\partial f}{\partial y}=\frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}-\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}$$

$$\frac{df}{dt} = v\frac{\partial f}{\partial y}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + w\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$+ f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + v\frac{\partial f}{\partial y} = \frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)$$

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} + \zeta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$+ f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{df}{dt} = \frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)$$

$$\frac{d\zeta}{dt}+\zeta\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+\frac{df}{dt}$$

$$=\frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}-\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)$$

$$\underbrace{\frac{d}{dt}(\zeta+f)}_{A}=\underbrace{-(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}_{B}\underbrace{-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)}_{C}+\underbrace{\frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}-\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)}_{D}$$

vorticity equation

A: Rate of change of absolute vorticity following the fluid motion

B: Effect of horizontal velocity divergence on vorticity (or vortex stretching term)

C: Transfer of vorticity between horizontal and vertical components ("twisting term" or "tilting term")

D: Effects of baroclinicity ("solenoidal term")

Term A: Rate of change of absolute vorticity following the fluid motion

$$\frac{d(\zeta + f)}{dt} = \frac{\partial(\zeta + f)}{\partial t} + u\frac{\partial(\zeta + f)}{\partial x} + v\frac{\partial(\zeta + f)}{\partial y} + w\frac{\partial(\zeta + f)}{\partial z}$$

local tendency of absolute vorticity

horizontal advection of absolute vorticity

vertical advection of absolute vorticity

Term B: Effect of horizontal velocity divergence on vorticity

$$-(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

The concentration or dilution of vorticity by the divergence field

This term is the fluid analog of the change in angular velocity resulting from a change in the moment of inertia of a solid body when angular momentum is conserved.

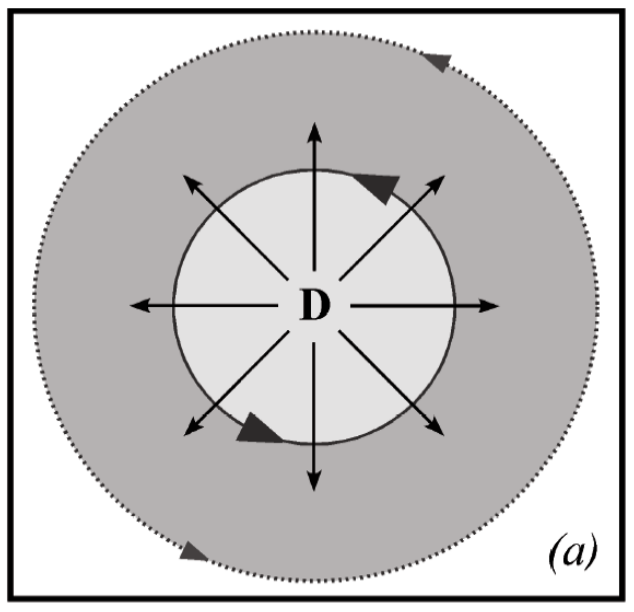
This mechanism for changing vorticity following the motion is very important in synoptic-scale midlatitude systems.

$$-(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

If $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) > 0$ (divergence),

then vorticity will decrease if absolute vorticity is positive.

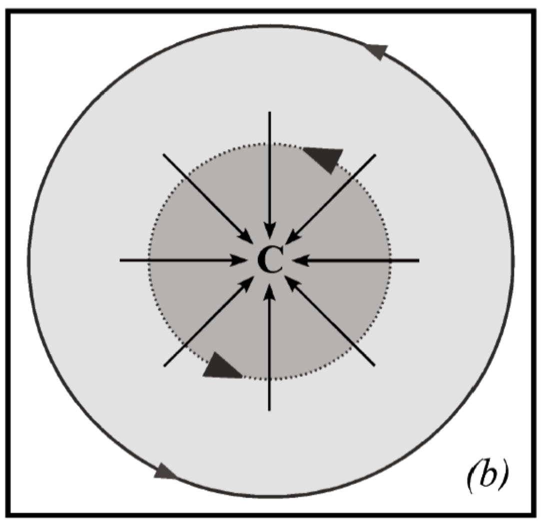
Vorticity will increase if absolute vorticity is negative.



If the horizontal flow is divergent, the area enclosed by a chain of fluid parcels will increase with time, and if circulation is to be conserved, the average absolute vorticity of the enclosed fluid must decrease (i.e., the vorticity will be diluted).

$$-(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

If $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) < 0$ (convergence),



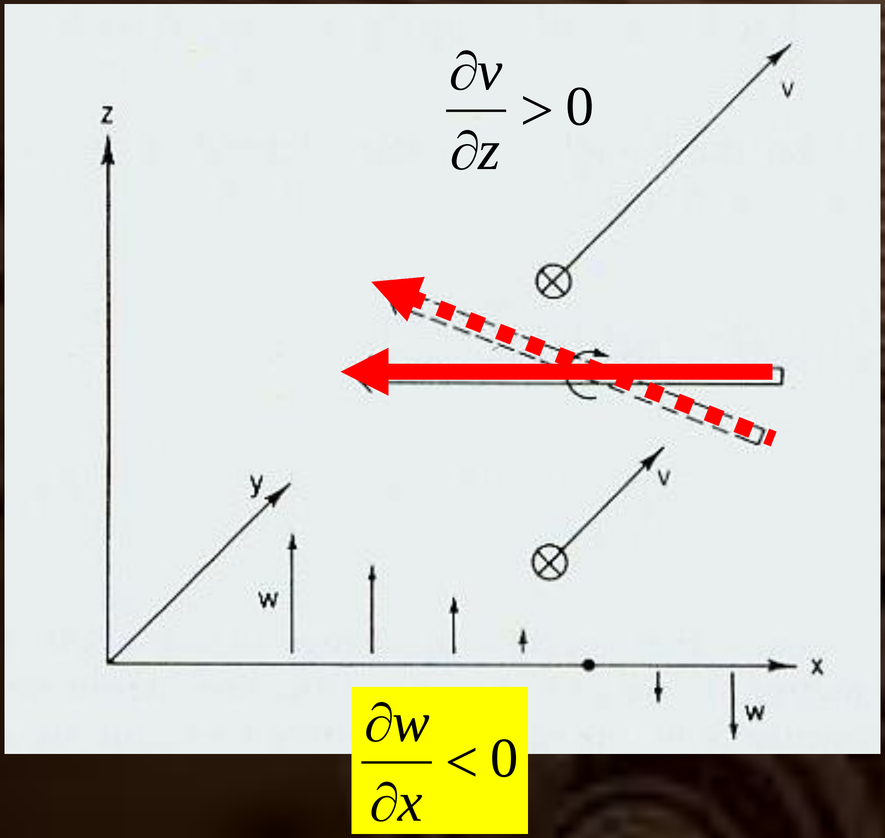
Then vorticity will increase if absolute vorticity is positive

Vorticity will decrease if absolute vorticity is negative.

If, however, the flow is convergent, the area enclosed by a chain of fluid parcels will decrease with time and the vorticity will be concentrated.

Term C: Transfer of vorticity between horizontal and vertical components ("twisting term" or "tilting term")

$$-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)$$

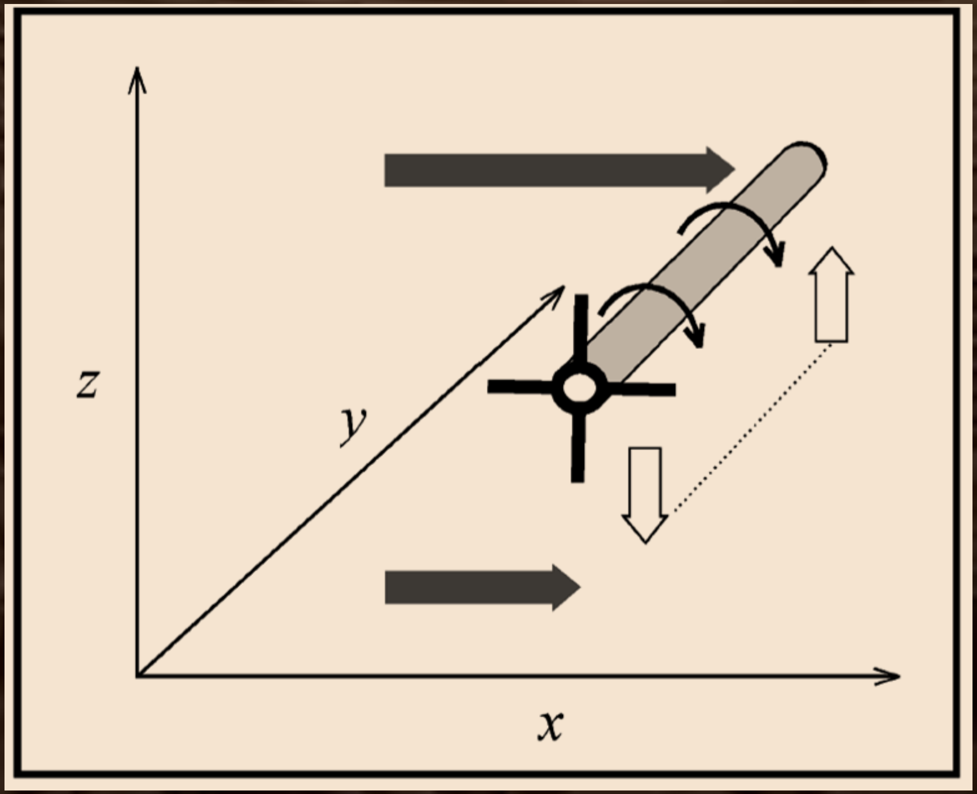


In this example, vertical shear of v-component wind is producing shear vorticity about an east-west axis. The orientation of the vorticity vector is shown by the solid red arrow.

East-west variations in the vertical velocity twist or tilt this "vortex tube" toward a more vertical orientation, as indicated by the broken red arrow. This gives the vorticity vector a component in the z-direction, indicating a transfer of vorticity from the horizontal to the vertical.

$$\frac{\partial v}{\partial z}\frac{\partial w}{\partial x} < 0 \rightarrow \frac{d(\zeta + f)}{dt} > 0$$

$$-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)$$



This term represents vertical vorticity generated by the tilting of horizontally oriented components of vorticity into the vertical by a nonuniform vertical motion field.

This mechanism is illustrated in which shows a region where the y component of velocity is increasing with height so that there is a component of shear vorticity oriented in the negative x direction as indicated by the double arrow.

If at the same time there is a vertical motion field in which w decreases with increasing x, advection by the vertical motion will tend to tilt the vorticity vector initially oriented parallel to x so that it has a component in the vertical.

Thus, if $\partial v / \partial z > 0$ and $\partial w / \partial x < 0$, there will be a generation of positive vertical vorticity.

Finally, the third term on the right in is just the microscopic equivalent of the solenoidal term in the circulation theorem (4.5).

To show this equivalence, we may apply Stokes's theorem to the solenoidal term to get

$$-\oint \alpha dp \equiv -\oint \alpha \nabla p \,.\, d\vec{l} = -\iint_A \nabla \times (\alpha \nabla p) \,.\, \vec{k} dA$$

where A is the horizontal area bounded by the curve l. Applying the vector identity $\nabla \times (\alpha \nabla p) \equiv \nabla \alpha \times \nabla p$ the equation becomes

$$-\oint \alpha dp = -\iint_A (\nabla \alpha \times \nabla p) \,.\, \vec{k} dA$$

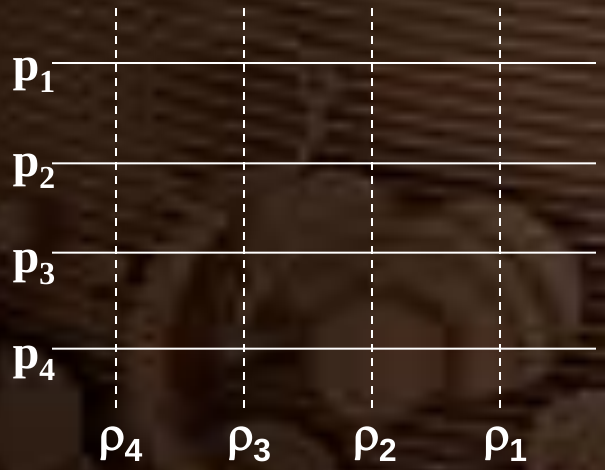
However, the solenoidal term in the vorticity equation can be written

$$-\left(\frac{\partial\alpha}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial\alpha}{\partial y}\frac{\partial p}{\partial x}\right)=-(\nabla\alpha\times\nabla p)\ .\ \vec{k}$$

Thus, the solenoidal term in the vorticity equation is just the limit of the solenoidalterm in the circulation theorem divided by the area when the area goes to zero.

Term D: Effects of baroclinicity ("solenoidal term")

$$\frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}-\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)$$



$p_4 > p_3 > p_2 > p_1$

$\rho_4 > \rho_3 > \rho_2 > \rho_1$

This term arises because of the horizontal variations in density that occur in a baroclinic atmosphere.

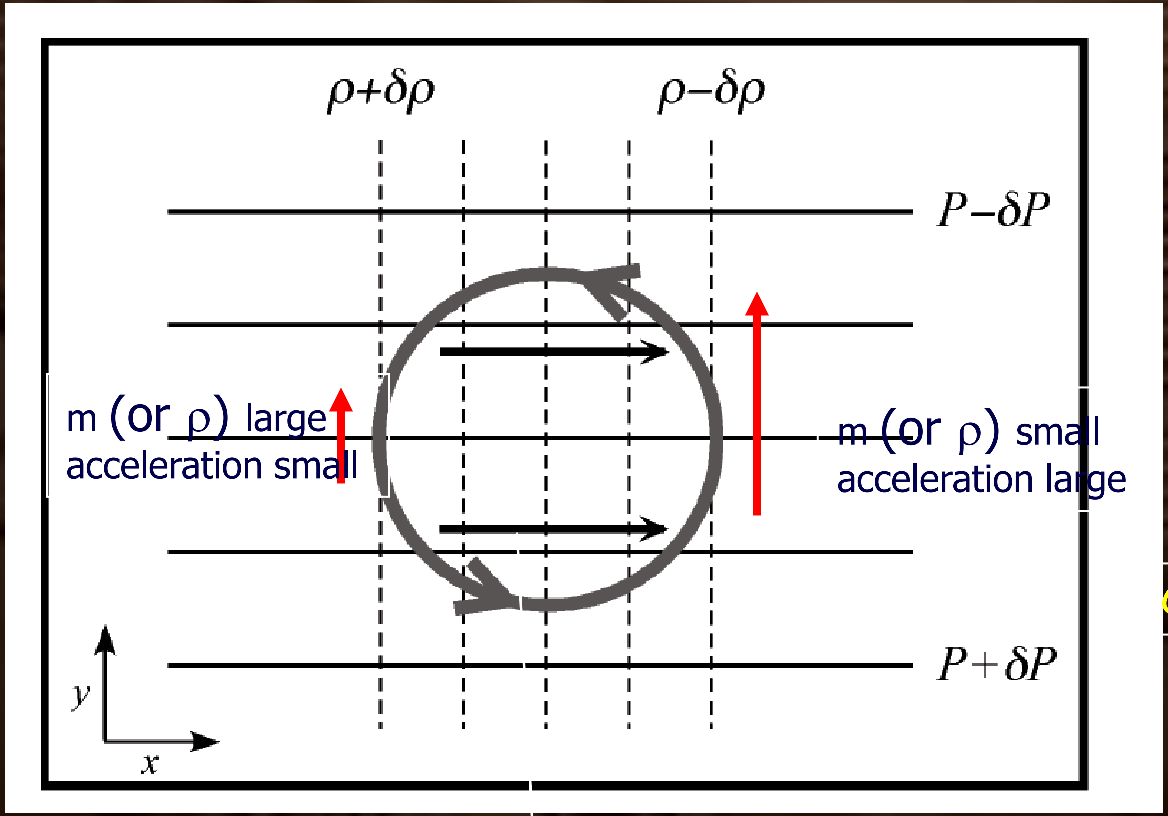
In this example, even though the pressure gradient is uniform, variations in density produce small variations in the pressure gradient force.

The variations in acceleration that result lead to the production of positive vorticity.

$$\frac{\partial p}{\partial y} < 0; \frac{\partial \rho}{\partial x} < 0 \rightarrow \frac{d(\zeta + f)}{dt} > 0$$

Solenoid: field loop that converts potential energy to kinetic energy

$$F = ma \qquad \frac{PGF}{m} = a \qquad \frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)$$



Cold advection pattern

geostrophic wind