دانشگاه رازی

Dynamic Meteorology 2

Lecture 13

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Waves are moving energy

Forces cause waves to move along air/water or within water

Wind (most surface ocean waves)

Movement of fluids with different densities

Internal waves often larger than surface waves

Mass movement into ocean

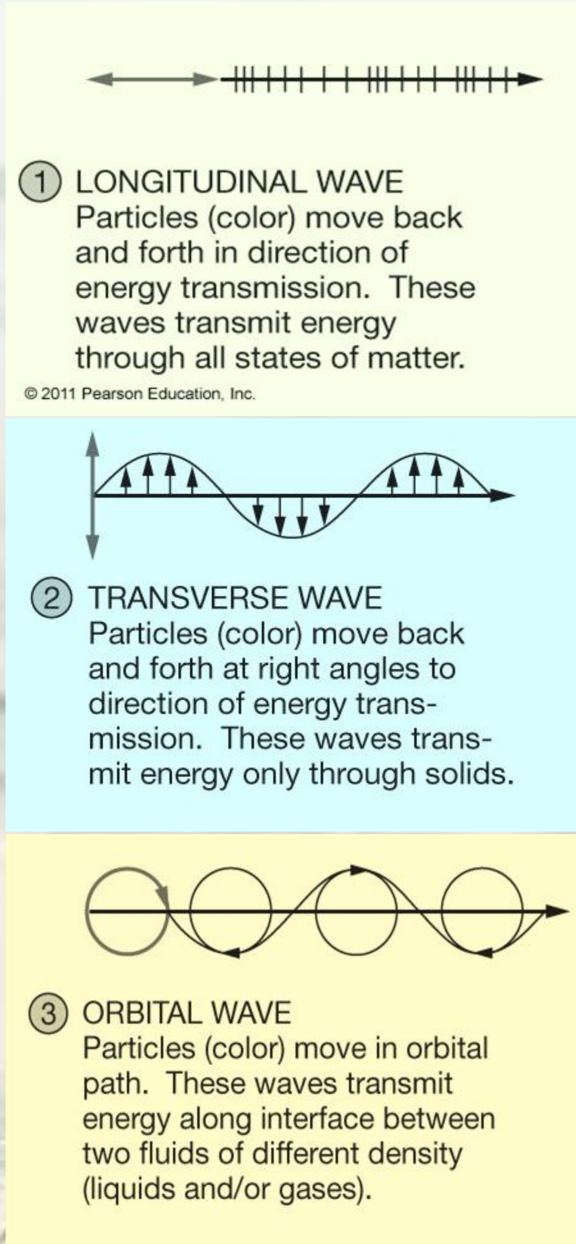
Splash waves

In an ocean wave, energy is moving at the speed of the wave, but water is not!

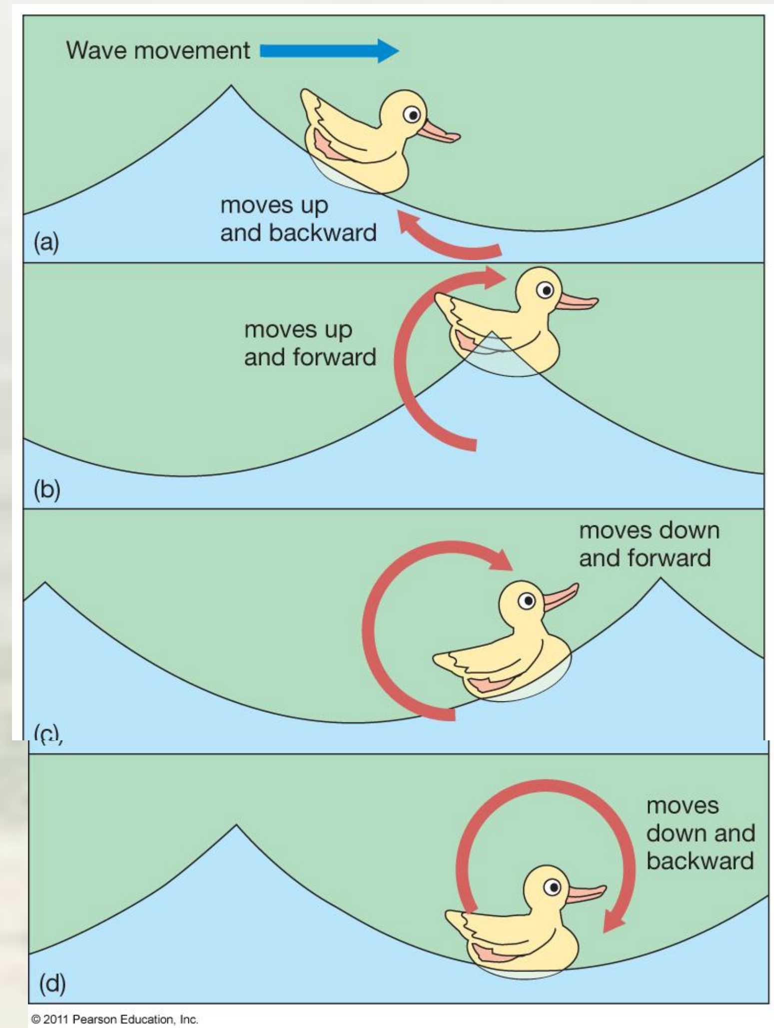
Waves move energy, with very little movement of particles (including water particles!)

The water 'associated' with a wave does not move continuously across the sea surface!

Types of waves



Wave particles move in a circle

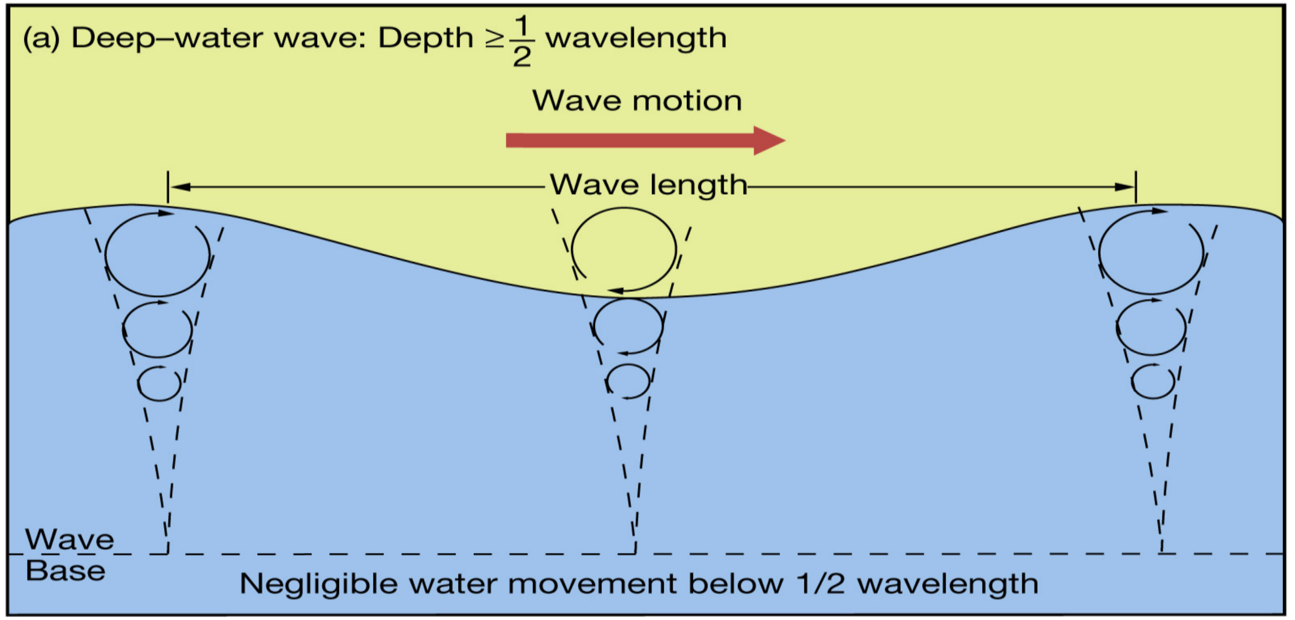


Deep Water Wave

Wave base is 1/2 wave length

Negligible water movement due to waves below this depth

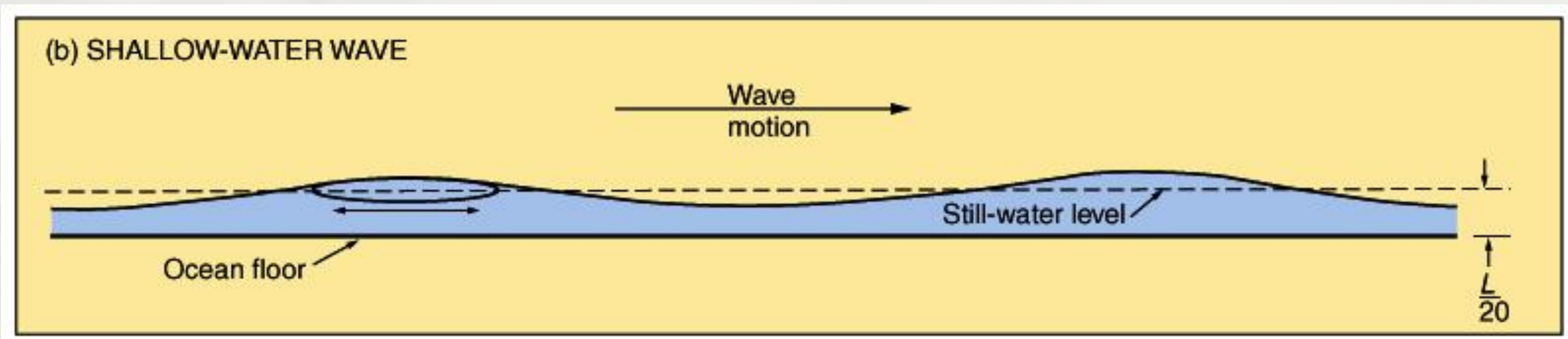
Wave speed form (celerity) is proportional to wavelength



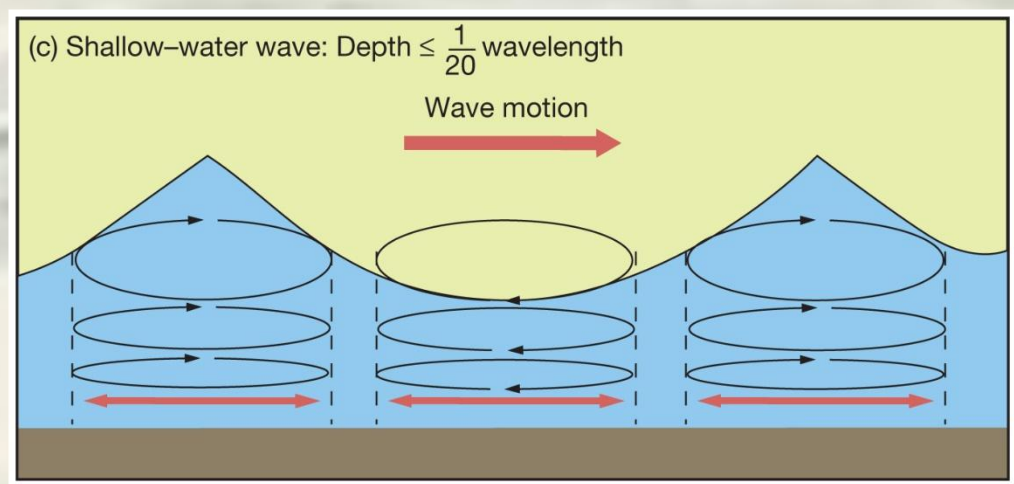
Shallow-water wave

Water depth is less than 1/20 wavelength

Wave speed (celerity) is proportional to depth of water



Orbital motion is flattened



Shallow-Water Wave

The second example of pure wave motion concerns the horizontally propagating oscillations known as shallow water waves.

1- Incompressible and homogeneous flow, where ρ_0 is a constant density => no sound waves (simplifies equations)

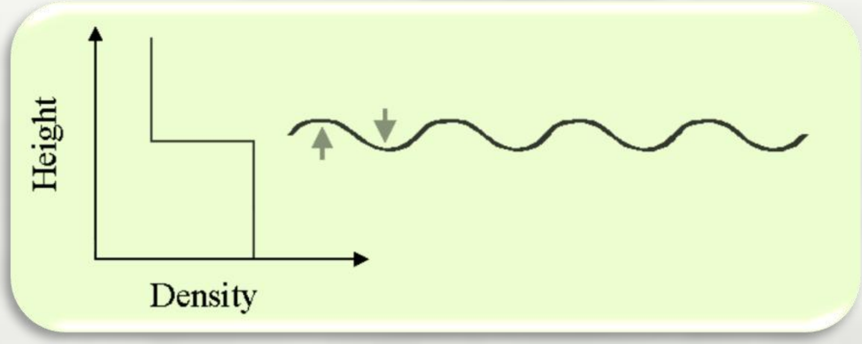
2- The flow is assumed to be inviscid

3-The water is so shallow that the flow velocity, $V(x, y)$, is constant with depth.

4- Shallow water, Require $\lambda_x \gg h$. Otherwise too deep for hydrostatic assumption.

$$\frac{\partial p}{\partial z} = \rho_0 g$$ hydrostatic approximation

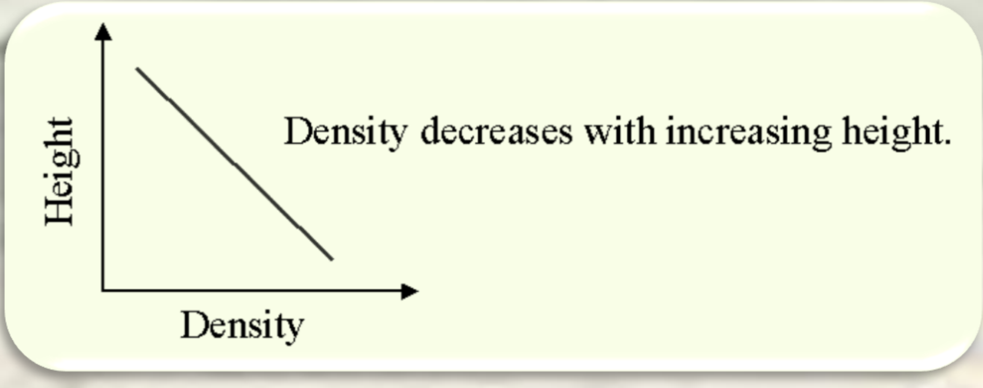
Restoring Force: Gravity



Force is transverse to direction of propagation

Gravity waves are buoyancy waves, the restoring force comes from Archimedes's principle.

They involve vertical displacement of air parcels, along slanted paths



They are found everywhere in the atmosphere

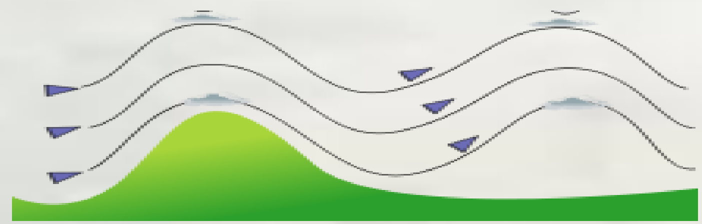
They can propagate vertically and horizontally, transporting momentum from their source to their sink

They are difficult features to represent correctly in global models, this is an area of active current research.

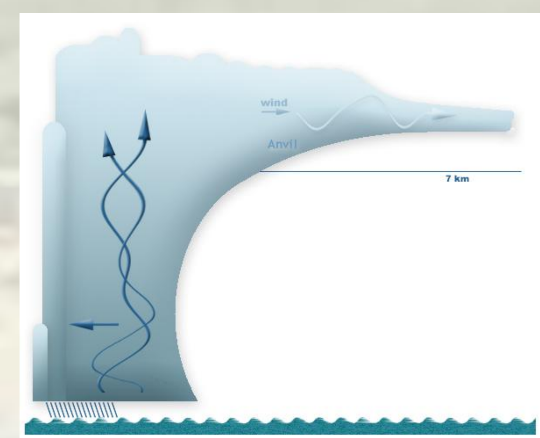
What causes them?

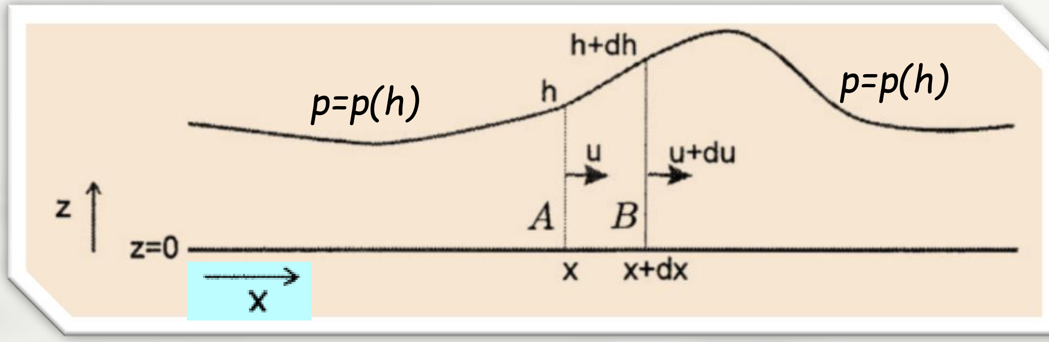
These waves can be generated in a variety of ways, but flow over mountains is one very common production method. This is shown in the next diagram.

Flow over a mountain range



Flow over convective cloud





Consider the volume of water bounded by vertical surfaces A and B in Figure. These surfaces are located at x and x+dx respectively.

We will now eliminate the vertical velocity w, thereby reducing the system to three equations for three variables.

z = h(x, y), the height of the free surface at point (x,y) that, the pressure is equal to atmospheric pressure p(h), assumed constant and uniform.

Integrating the hydrostatic equation over the depth of the fluid, *h(x, y)*, gives the pressure between *z and h* below the surface

$$\int_z^h \frac{\partial p}{\partial z} dz + \int_z^h \rho_0 g dz = 0 \qquad p(z) = p(h) + \rho_0 g(h - z) \quad *$$

where p(h) is the pressure at the top of the layer of shallow water due to the layer above, which we take to be a constant.

We assume that

V is initially a function of (x, y) only, and since h is a function of (x, y).

This equation indicates that ***V*** will remain two-dimensional for all time.

Using * to replace pressure in the momentum equation gives

$$\frac{d_h \vec{V}}{dt} = -g\nabla_h h - f\hat{k} \times \vec{V} \quad *'$$

$$-\frac{1}{\rho}\nabla p = -g\nabla_h h$$

$$\frac{d_h\vec{V}}{dt} = -g\nabla_h h - f\hat{k}\times\vec{V} \quad *'$$

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} - fv + g\dfrac{\partial h}{\partial x} = 0 \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + fu + g\dfrac{\partial h}{\partial y} = 0 \\ \dfrac{\partial h}{\partial t} + u\dfrac{\partial h}{\partial x} + v\dfrac{\partial h}{\partial y} + h(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}) = 0 \end{cases}$$

Mass conservation for a constant density flow has the simple form

$$\nabla.(u, v, w) = 0$$

Next, we integrate the continuity equation through the full depth of the fluid. Since u and v are constant with z,

$$\int_0^h \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz = h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

The third term of the continuity equation integrates to

$$\int_0^h \left(\frac{\partial w}{\partial z}\right) dz = w(h) - w(0) = \frac{dh}{dt}$$

Here we have assumed that the bottom boundary is flat, so that the vertical velocity there vanishes: $w(0) = 0$.

$$\frac{dh}{dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Water pressure is a function of density and temperature, $p = f(T, \rho)$ but following the motion incompressibility implies

$$dp = \frac{\partial f}{\partial T} dT + \frac{\partial f}{\partial \rho} d\rho = 0$$

$$dp / dt = 0$$

$$p(z) = \rho_0 g(h - z) + p(h)$$

$$\frac{d_h h}{dt} = \frac{dz}{dt} = w(h)$$

$w = \dfrac{dz}{dt}$ is the vertical motion, which is a function of (x, y, z)

Integrating $\nabla.(u, v, w) = 0$ $w(h) = -h\nabla_h.\vec{V}$

$$\frac{d_h h}{dt} = -h\nabla_h.\vec{V} \quad *''$$

The shallow water equations consist of 2 and 3 star.

The linearized momentum equation *' mass continuity equations *" are:

$$\frac{\partial u'}{\partial t} = -g\frac{\partial h'}{\partial x} + fv'$$

$$\frac{\partial v'}{\partial t} = -g\frac{\partial h'}{\partial y} - fu'$$

$$\frac{\partial h'}{\partial t} = -\bar{h}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right)$$

Primes denote perturbation values, that is, departures from the state of rest.

These equations represent a set of three coupled first-order partial differential equations in the unknown u', v', h'

One approach to solving these equations is to form and solve a single third-order partial differential equation.

This is accomplished by taking $\frac{\partial}{\partial t}$ *of the third equation,* which gives

$$\frac{\partial^2 h'}{\partial t^2} = -\bar{h}\left(\frac{\partial^2 u'}{\partial t \partial x} + \frac{\partial^2 v'}{\partial t \partial y}\right)$$

The terms on the right side of this equation may be replaced using the first two equations of

$$\frac{\partial^2 h'}{\partial t^2} = -\bar{h}(g\nabla_h^2 h' - f\zeta')$$

Again take $\frac{\partial}{\partial t}$ *which gives the third-order equation*

$$\frac{\partial^3 h'}{\partial t^3} + (f^2 - g\bar{h}\nabla_h^2)\frac{\partial h'}{\partial t} = 0 \quad *$$

where $\partial\zeta' / \partial t$ is replaced using the linearized version of

$$\frac{d_h}{dt}(\zeta + f) = -(\zeta + f)(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$$

(noting that f is taken constant here):

$$\frac{\partial \zeta'}{\partial t} = -f(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y})$$

Assuming that the lateral boundaries are periodic and, since the coefficients f *and* $g\bar{h}$ constant, we may assume wave solutions of the form

$$h' = \mathrm{Re}\left\{ A e^{i(kx+ly-\omega t)} \right\} \quad *'$$

Using *' in *gives a cubic polynomial for the frequency

$$\omega^3 - \omega\left[f^2 + g\bar{h}(k^2 + l^2) \right] = 0$$

This is the dispersion relationship for shallow water waves.

Clearly $\omega = 0$ is a solution, and if $\omega \neq 0$, then

$$\omega^2 = f^2 + g\bar{h}(k^2 + l^2)$$

For readers familiar with linear algebra, we note an alternative solution method.

The solution of the form $h' = \mathrm{Re}\left\{Ae^{i(kx+ly-\omega t)}\right\}$ for h', u', and v', and substituting directly into

$$\frac{\partial u'}{\partial t} = -g\frac{\partial h'}{\partial x} + fv' \qquad \frac{\partial v'}{\partial t} = -g\frac{\partial h'}{\partial y} - fu' \qquad \frac{\partial h'}{\partial t} = -\bar{h}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right)$$

converts the set of partial differential equations to algebraic equations that may be written as

$$Ax = 0$$

where x is the column vector of the unknowns $\begin{bmatrix} u' \\ v' \\ h' \end{bmatrix}$

$$\begin{bmatrix} -i\omega & -f & ikg \\ f & -iv & il\,g \\ \bar{h}k & -\bar{h}l & -\omega \end{bmatrix} \begin{bmatrix} u' \\ v' \\ h' \end{bmatrix} = 0$$

A nontrivial solution to is obtained only if A is not invertible.

This is enforced by setting the determinant of A to zero, which gives

$$\omega^3 - \omega\left[f^2 + g\bar{h}(k^2 + l^2) \right] = 0$$

Geostrophic Balance

Turning now to the structure of the waves, consider first the case where **ω=0**. Appealing to

$$\frac{\partial u'}{\partial t} = -g\frac{\partial h'}{\partial x} + fv'$$

$$\frac{\partial v'}{\partial t} = -g\frac{\partial h'}{\partial y} - fu'$$

$$\frac{\partial h'}{\partial t} = -\bar{h}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right)$$

we see that since the waves are stationary, the left sides are zero.

Clearly, these waves depend on rotation, since when f = 0, the wave amplitude, A, must also be zero.

For non zero f :

$$v' = \frac{g}{f}\frac{\partial h'}{\partial x} \qquad\qquad u' = -\frac{g}{f}\frac{\partial h'}{\partial y}$$

The stationary waves are in a state of geostrophic balance and are an example of Rossby waves in the limit of constant Coriolis parameter.

inertia-gravity waves

Solutions for $\omega \neq 0$ are called inertia-gravity waves, since particle oscillations depend on both gravitational and inertial forces.

So that we may focus here on the gravitational aspect.

In the limiting case f=0 , we have simply gravity waves, and from

$$\omega^2 = f^2 + g\bar{h}(k^2 + l^2)$$

The shallow water gravity wave speed becomes

$$c = \sqrt{g\bar{h}}$$

$$c = \sqrt{g\bar{h}}$$

Since the waves all move at the same speed, they are nondispersive.

$$For \;\; \bar{h} = 4 \; km \; deep \quad \rightarrow c \approx 200 \; ms^{-1}$$

Thus, long waves on the ocean surface travel very rapidly.