Dynamic Meteorology 2

Lecture 11

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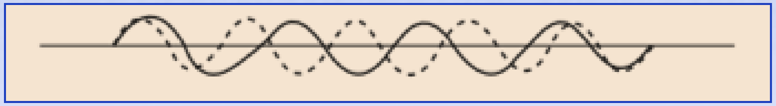
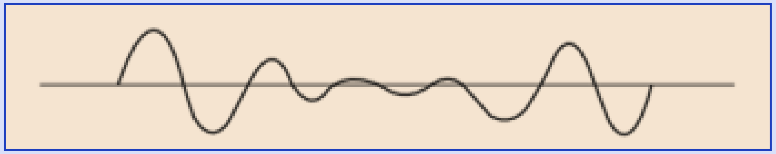
Dispersion and Group Velocity

Wave groups formed from two sinusoidal components of slightly different wavelengths.

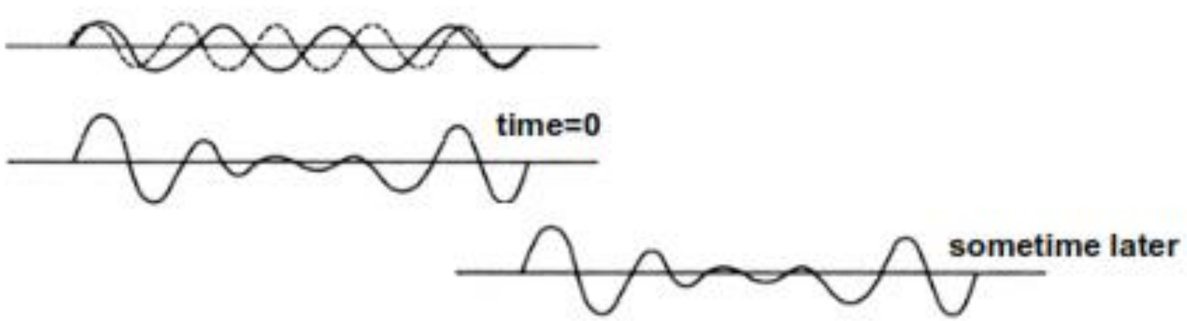


For nondispersive waves, propagates without change of shape.



For dispersive waves, the shape of the pattern changes in time.

Non-Dispersive Waves



- Some types of waves, such as acoustic waves, have phase speeds that are independent of the wave number.
- In such *nondispersive waves a spatially* localized disturbance consisting of a number of Fourier wave components (a *wave group) will preserve its shape as it propagates in space at the phase speed of the* wave.

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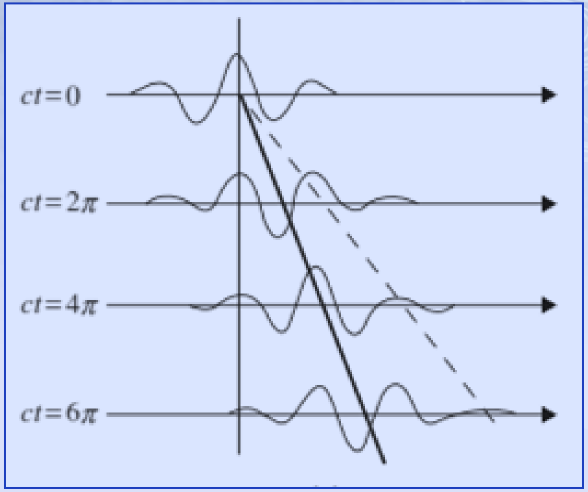
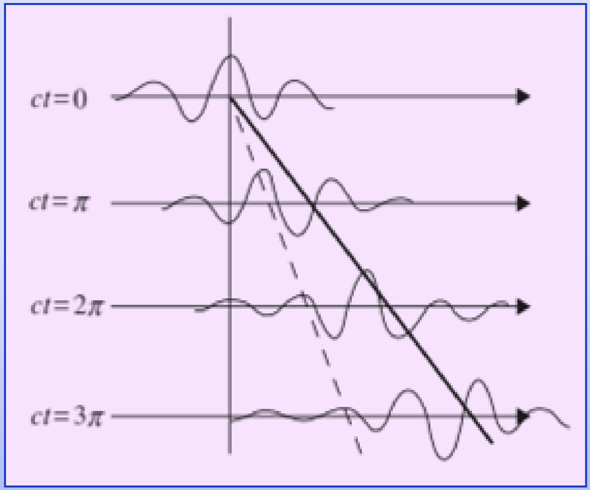
Schematic showing propagation of wave groups

group velocity less than phase speed

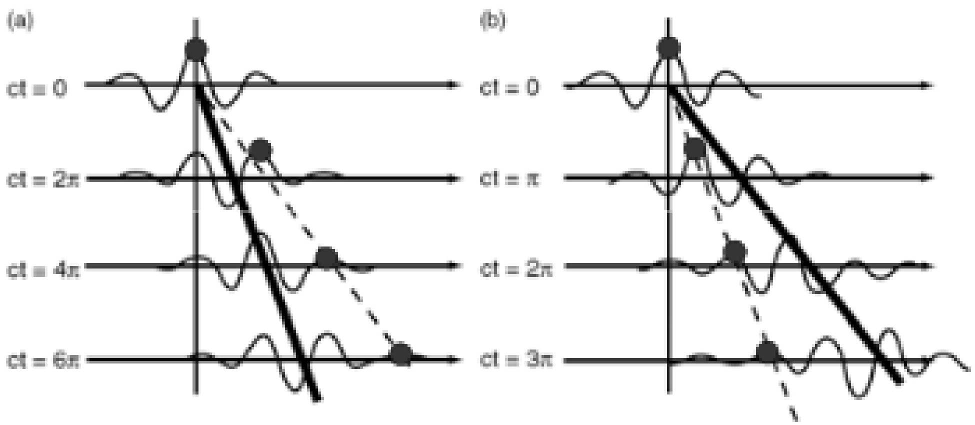


group velocity greater than phase speed



Heavy lines show group velocity, and light lines show phase speed

Dispersive Waves



- For dispersive waves, the shape of a wave group will not remain constant as the group propagates.
- Furthermore, the group generally broadens in the course of time, that is, the energy is *dispersed.*
- When waves are dispersive, the speed of the wave group is generally different from the average phase speed of the individual Fourier components.
- In synoptic-scale atmospheric disturbances, however, the group velocity exceeds the phase velocity.

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Usually every spectral component propagates with its own phase speed.

This leads to dispersion of a wave package.

A wave package consisting of components around k and ω propagates with the group velocity $c_g = \frac{\partial \omega}{\partial k}$.

An expression for group velocity, which is the velocity at which the observable disturbance (and thus the energy) propagates, can be derived as follows:

We consider the superposition of two horizontally propagating waves of equal amplitude but slightly different wavelengths, with wave numbers and frequencies differing by $2\delta k$ and $2\delta\omega$ respectively.

Assume the sum of two waves of equal amplitude, and

$$k_1 = k + \delta k, \qquad k_2 = k - \delta k, \qquad \omega_1 = \omega + \delta\omega, \qquad \omega_2 = \omega - \delta\omega$$

The total disturbance is:

$$\psi(x,t) = e^{i\left[(k+\delta k)x-(\omega+\delta\omega)t\right]} + e^{i\left[(k-\delta k)x-(\omega-\delta\omega)t\right]}$$

$$\psi(x,t) = \left[e^{i(\delta k\,x-\delta\omega\,t)} + e^{-i(\delta k\,x-\delta\omega\,t)}\right]e^{i(kx-\omega t)}$$

Rearranging terms and applying the Euler formula gives

$$\psi(x,t) = 2\cos\left(\delta k\,x - \delta\omega\,t\right)\mathrm{e}^{i(kx-\omega t)}$$

This disturbance is the product of a high-frequency carrier wave of wavelength $2\pi/k$ whose phase speed, ω/k,

is the average for the two Fourier components, and a low-frequency envelope of wavelength $2\pi/\delta k$ that travels at the speed $\delta\omega/\delta k$

The envelope has a wavelength of $\lambda_g = \dfrac{2\pi}{\delta k}$

The envelope has a phase $\phi_g = \delta k\, x - \delta\omega t$

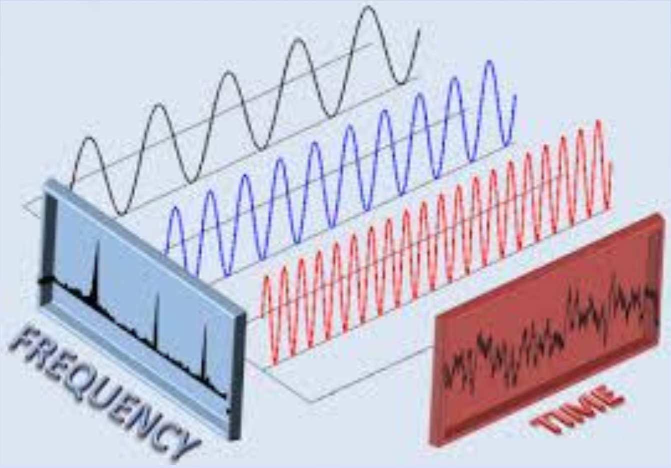
$$c_{gx} = \lim_{\delta k \to 0} \frac{\delta\omega}{\delta k} = \frac{\partial\omega}{\partial k}$$ group velocity

The wave energy thus propagates at the group velocity.

This result applies generally to arbitrary wave envelopes provided that the wavelength of the wave group, $2\pi/\delta k$, *is large compared to the wavelength of the dominant component*, $2\pi/k$.

Fourier Series

Each wave package (disturbance) can be represented as a sum of waves.



Fourier Series

The representation of a perturbation as a simple sinusoidal wave might seem an oversimplification, since disturbances in the atmosphere are never purely sinusoidal.

It can be shown, however, that any reasonably well-behaved function of longitude can be represented in terms of a zonal mean plus a *Fourier series* of sinusoidal components:

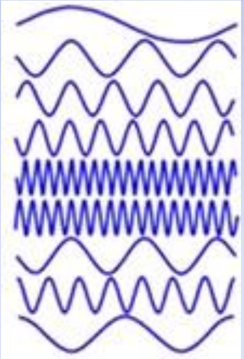
$$f(x) = \sum_{s=1}^{\infty} (A_s \sin k_s x + B_s \cos k_s x)$$

$$k_s = \frac{2\pi s}{L}$$ The zonal wave number (m^{-1})

L is the distance around a latitude circle,

s, the planetary wave number, is an integer des *designating* the number of waves around a latitude circle.

Fourier Series



$$f(x) = \sum_{s=1}^{\infty} (A_s \sin k_s x + B_s \cos k_s x)$$

disturbance

The coefficients A_s are calculated by multiplying both sides of equ. by

$\sin\left(\frac{2\pi nx}{L}\right)$ where *n is an* integer, and integrating around a latitude circle.

Applying the orthogonality relationships

$$\int_0^L \sin\frac{2\pi sx}{L}\sin\frac{2\pi nx}{L}dx = \begin{cases} 0 & s \neq n \\ L/2 & s = n \end{cases}$$

$$\therefore A_s = \frac{2}{L}\int_0^L f(x)\sin\frac{2\pi sx}{L}dx$$

In a similar fashion, multiplying both sides in equ. by $\cos\left(\frac{2\pi nx}{L}\right)$ and integrating gives:

$$B_s = \frac{2}{L}\int_0^L f(x)\cos\frac{2\pi sx}{L}dx$$

A_s and B_s are called the *Fourier coefficients* *and* $f_s(x) = A_s \sin k_s x + B_s \cos k_s x$

is called the s^{th} Fourier component or s^{th} harmonic of the function f (x)

The expression for a Fourier component may be written more compactly using complex exponential notation.

According to the Euler formula, $e^{i\phi} = \cos\phi + i\sin\phi \qquad i \equiv (-1)^{1/2}$

$$f_s(x) = \mathrm{Re}(C_s e^{ik_s x}) = \mathrm{Re}(C_s \cos k_s x + iC_s \sin k_s x)$$

C_s is a complex coefficient

Comparing $B_s = \mathrm{Re}(C_s) \qquad A_s = -\mathrm{Im}(C_s)$

Wave Properties in Two and Three Dimensions

A two-dimensional plane wave in a scalar field,f, may be expressed as

$$f(x,y,t) = \mathrm{Re}(Ae^{i(kx+ly-\omega t)}) = \mathrm{Re}(Ae^{i\phi})$$

Independent variables (*x, y) and t represent space and time, respectively.*

$\phi = kx + ly + C$ is uniform along lines of constant $kx + ly$

C constant

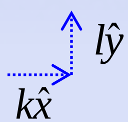
This implies $d\phi = \frac{\partial \phi}{\partial x}\delta x + \frac{\partial \phi}{\partial y}\delta y = 0$ for ϕ constant

Therefore, the slope of these lines is $\left.\frac{\delta y}{\delta x}\right|_{\phi} = -k / l$

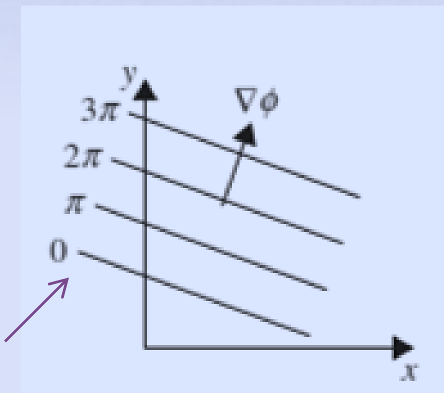
Constant values of $e^{i\phi} = e^{i(\phi+2\pi n)}$

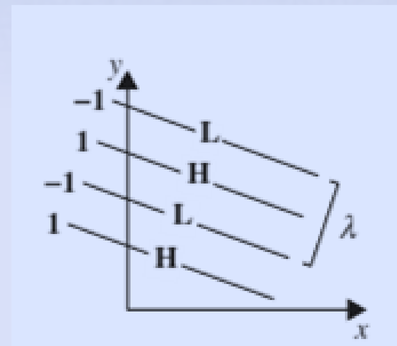
where n is an integer, define lines of constant phase, such as lows and highs.

Two-dimensional plane wave at a fixed time:



Lines of constant ϕ





$$f = kx + ly - nt$$

$$\vec{K} = \nabla\phi = k\hat{x} + l\hat{y}$$

$$\vec{K} \wedge const. f$$

phase angle, ϕ ,

phase $e^{i\phi}$

Wavelength is denoted λ. Note that if ω > 0, the wave travels in the direction of the wave vector, $\nabla\phi$

The wave vector is defined by $\vec{K} = \nabla\phi$

|K| =(phase-change)/(unit-length)

k = (phase-change in x-direction)/(unit-length)

l = (phase-change in y-direction)/(unit-length)

$k = |\vec{K}|$ *is the total wave number,*

and therefore $\lambda = \dfrac{2\pi}{k}$ is the *wavelength that is, the distance between lines of constant phase.*

At any fixed point in space, $\phi = C - \omega t$, C constant, so ϕ is a linear function of time.

Therefore, frequency is defined as $\omega = -\dfrac{\partial \phi}{\partial t}$

which is the rate at which lines of constant phase pass a fixed point in space.

The wave *period is* $\dfrac{2\pi}{|\omega|}$, which is the length of time between points of constant phase (units: seconds).

The wave *phase speed* ($m\ s^{-1}$) is determined by how fast lines of constant phase move along the wave vector

$$c = \frac{\omega}{k} = -\frac{1}{|\nabla \phi|}\frac{\partial \phi}{\partial t}$$

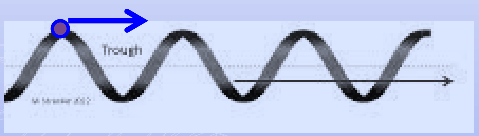
In particular, for two and three dimensions, the phase speed is not the same as

$$c_x = \frac{\omega}{k} \qquad\qquad c_y = \frac{\omega}{l}$$

Which define the rate at which phase lines travel along the x and y coordinate axes, respectively.

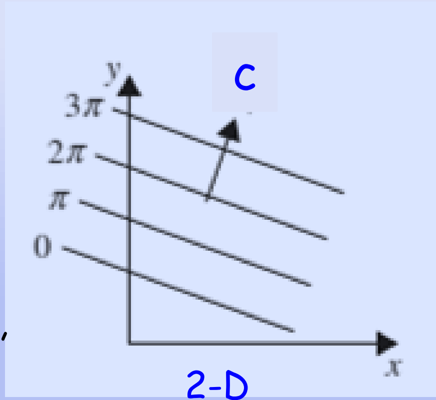
Furthermore, c, c_x, and c_y do not satisfy rules of vector addition, so

$$c^2 \neq c_x^2 + c_y^2$$



1-D case: Move with point of constant phase - e.g., crest

By analogy with 1-D, for phase speed c, perpendicular to lines of constant ϕ:



2-D

Phase Speed in Coordinate Directions

Move with point of constant phase - e.g., crest $\frac{d\phi}{dt} = 0$

move only in x-direction: $\frac{\partial \phi}{\partial t} + \left(\frac{dx}{dt}\right)\frac{\partial \phi}{\partial x} = 0$

$$\frac{dx}{dt} = c_x = -\frac{\partial \phi / \partial t}{\partial \phi / \partial x} = \frac{\omega}{k}$$

Similarly, looking at phase change only in y direction (e.g., crest movement in y)

$$\frac{dy}{dt} = c_y = -\frac{\partial \phi / \partial t}{\partial \phi / \partial y} = \frac{\omega}{l}$$

$$\begin{cases} \vec{K} = \nabla \phi & \text{wave vector} \\ \omega = -\dfrac{\partial \phi}{\partial t} & \text{frequency} \end{cases}$$

$$\frac{\partial \vec{K}}{\partial t} + \nabla \omega = 0 \quad *$$

Frequency is defined by a dispersion relationship, so that ω is a function of **K**.

As such, $\nabla \omega$ may be evaluated using the chain rule

$$\nabla \omega = \nabla_k \omega . \nabla) \vec{K}$$

Where $\nabla_k \omega = (\dfrac{\partial \omega}{\partial k}, \dfrac{\partial \omega}{\partial l})$ is the group velocity,

Therefore, * can be written as $$\frac{\partial \vec{K}}{\partial t} + (C_g . \nabla) \vec{K} = 0$$

Consequently, in a frame of reference moving with the group velocity, the wave vector is conserved; that is, we follow a group of waves with fixed wavelength and frequency.