Dynamic Meteorology 2

Lecture ١٠

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Chapter ۵
Atmospheric Oscillations
Linear Perturbation Theory

To describe large-scale atmospheric motions with some accuracy requires numerical techniques.

This makes it difficult to understand the fundamental processes and the balances of forces.

Meteorological disturbances often have a wave-like character.

Do discuss waves in the atmosphere or in fluids, we linearize the governing equations using the perturbation method.

Governing equations of the atmosphere are non-linear (e.g. advection terms) and cannot be solved analytically in general!

Perturbation Method

We linearize the equations and study here the linear wave solutions analytically.

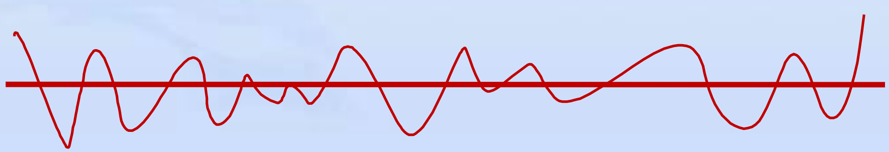
In the perturbation method, all field variables are divided into two parts:

1) a basic state portion, which is usually assumed to be independent of time and longitude,

2) a perturbation portion, which is the local deviation of the field from the basic state.

Field variable = (basic state part) + (pertubation part)

Assume $\overline{u}$, $\overline{v}$, $\overline{p}$, $\overline{T}$, etc. are time and longitude-averaged variables,



$$u(x,t) = \overline{u} + u'(x,t)$$

In that case, for example, the inertial acceleration $u\frac{\partial u}{\partial x}$ can be written:

$$u\frac{\partial u}{\partial x}=\left(\bar{u}+u'\right)\frac{\partial\left(\bar{u}+u'\right)}{\partial x}=\bar{u}\frac{\partial u'}{\partial x}+u'\frac{\partial u'}{\partial x}=\bar{u}\frac{\partial u'}{\partial x}$$

$$if \quad \left|\frac{u'}{\bar{u}}\right| = 1 \qquad or \qquad \left|u'\right| = \left|\bar{u}\right|$$

$$then \quad \left|\bar{u}\frac{\partial u'}{\partial x}\right| ? \left|u'\frac{\partial u'}{\partial x}\right|$$

Non-linear equations are reduced to linear differential equations in the perturbation variables in which the basic state variables are specified coefficients.

Basic assumptions of perturbation theory are:

1) The basic state variables must themselves satisfy the governing equations.

$\bar{\rho}$, $\bar{p}$ define the basic atmospheric state and satisfy $$\frac{\partial \bar{p}}{\partial z} = -g\bar{\rho}$$

2) Perturbations must be small enough to neglect all products of perturbations.

Applying the perturbation method to the *u*-momentum equation is illustrated below.

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv$$

$$\frac{dv}{dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{dw}{dt} + g = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

$$u(x,y,z,t) = \overline{u} + u'(x,y,z,t)$$

$$v(x,y,z,t) = \overline{v} + v'(x,y,z,t)$$

$$w(x,y,z,t) = w'(x,y,z,t); \quad assume\ \overline{w} = 0$$

$$p(x,y,z,t) = \overline{p}(x,y,z) + p'(x,y,z,t)$$

$$\rho(x,y,z,t) = \overline{\rho}(x,y,z) + \rho'(x,y,z,t)$$

$$\frac{\partial(\overline{u}+u')}{\partial t} + (\overline{u}+u')\frac{\partial(\overline{u}+u')}{\partial x} + (\overline{v}+v')\frac{\partial(\overline{u}+u')}{\partial y} + (\overline{w}+w')\frac{\partial(\overline{u}+u')}{\partial z} = -\frac{1}{(\overline{\rho}+\rho')}\frac{\partial(\overline{p}+p')}{\partial x} + f(\overline{v}+v')$$

We can simplify this equation by recognizing that the basic state variables are independent of time, and that only the pressure and density basic state variables are functions of *z*.

Also, we assumed that the base-state vertical velocity is zero. The equation then becomes

$$\frac{\partial u'}{\partial t}+(\bar{u}+u')\frac{\partial u'}{\partial x}+(\bar{v}+v')\frac{\partial u'}{\partial y}+w'\frac{\partial u'}{\partial z}=-\frac{1}{(\bar{\rho}+\rho')}\frac{\partial(\bar{p}+p')}{\partial x}+f(\bar{v}+v')$$

Since the perturbation quantities are very small, we assume that we can ignore products of perturbation quantities. This further simplifies the equation to

$$\frac{\partial u'}{\partial t}+\bar{u}\frac{\partial u'}{\partial x}+\bar{v}\frac{\partial u'}{\partial y}=-\frac{1}{(\bar{\rho}+\rho')}\frac{\partial(\bar{p}+p')}{\partial x}+f(\bar{v}+v')$$

We also assume that we can ignore perturbations of density in the horizontal pressure gradient term (similar to the Boussinesq approximation), to get

$$\frac{\partial u'}{\partial t}+\bar{u}\frac{\partial u'}{\partial x}+\bar{v}\frac{\partial u'}{\partial y}=-\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x}-\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}+f\bar{v}+fv'$$

And finally, if we assume that the basic state is in geostrophic balance, then

$$-\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x} + f\bar{v} = 0,$$

so that we are left with

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + \bar{v}\frac{\partial u'}{\partial y} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} + fv'$$

This is the linearized, or perturbation form of, the *u*-momentum equation.

Linearization of the *v*-momentum equation proceeds in a similar manner.

$$\frac{\partial v'}{\partial t} + \bar{u}\frac{\partial v'}{\partial x} + \bar{v}\frac{\partial v'}{\partial y} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial y} - fu'$$

LINEARIZING THE W-MOMENTUM EQUATION

The *w*-momentum equation is a bit trickier, because we can't ignore the density perturbation in the vertical pressure gradient term like we could in the horizontal pressure gradient term of the *u*-momentum equation. So, after substituting the basic state and perturbation variables into the *w*-momentum equation we get

$$\frac{\partial w'}{\partial t} + \bar{u}\frac{\partial w'}{\partial x} + \bar{v}\frac{\partial w'}{\partial y} = -\frac{1}{(\bar{\rho} + \rho')}\frac{\partial(\bar{p} + p')}{\partial z} - g$$

A rule of algebra tells us that if $a \ll 1$, then

$$\frac{1}{1+a} \cong 1 - a$$

Using this rule we can write

$$\frac{1}{\bar{\rho} + \rho'} = \frac{1}{\bar{\rho}(1 + \rho'/\bar{\rho})} \cong \frac{1}{\bar{\rho}}\left(1 - \frac{\rho'}{\bar{\rho}}\right)$$

Using this, the RHS of the *w*-momentum equation becomes

$$-\frac{1}{\bar{\rho}+\rho'}\frac{\partial}{\partial z}(\bar{p}+p')-g=\frac{1}{\bar{\rho}}\left(\frac{\rho'}{\bar{\rho}}-1\right)\frac{\partial}{\partial z}(\bar{p}+p')-g$$

$$=\frac{1}{\bar{\rho}}\left(\frac{\rho'}{\bar{\rho}}\frac{\partial\bar{p}}{\partial z}+\frac{\rho'}{\bar{\rho}}\frac{\partial p'}{\partial z}-\frac{\partial\bar{p}}{\partial z}-\frac{\partial p'}{\partial z}\right)-g$$

and since we can ignore products of perturbation terms, this simplifies to

$$\frac{1}{\bar{\rho}}\left(\frac{\rho'}{\bar{\rho}}\frac{\partial\bar{p}}{\partial z}-\frac{\partial\bar{p}}{\partial z}-\frac{\partial p'}{\partial z}\right)-g$$

If the basic state is in hydrostatic balance, then $\frac{\partial\bar{p}}{\partial z}=-\bar{\rho}\,g$

Substituting this into the equation above it gives

$$\frac{1}{\bar{\rho}}\left(\frac{\rho'}{\bar{\rho}}(-\bar{\rho}g)-(-\bar{\rho}g)-\frac{\partial p'}{\partial z}\right)-g=-\frac{\rho'}{\bar{\rho}}g-\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z}$$

so that the linearized *w*-momentum equation is

$$\frac{\partial w'}{\partial t} + \bar{u}\frac{\partial w'}{\partial x} + \bar{v}\frac{\partial w'}{\partial y} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z} - \frac{\rho'}{\bar{\rho}}g$$

Note that what we've done is to use the basic state density everywhere except in the buoyancy term (the term involving *g*), where we used the perturbation density.

This is essentially the Boussinesq approximation, the difference being that the reference density is allowed to vary spatially, whereas in the Boussinesq approximation the reference density is assumed to be a true constant.

THE FINAL FORM OF THE PERTURBATION EQUATIONS

If we assume that the basic state is in geostrophic and hydrostatic balance, and that the base-state density is a function of *z* only, the linearized momentum and continuity equations are

$$\frac{\partial u'}{\partial t}+\bar{u}\frac{\partial u'}{\partial x}+\bar{v}\frac{\partial u'}{\partial y}=-\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}+fv'$$

$$\frac{\partial v'}{\partial t}+\bar{u}\frac{\partial v'}{\partial x}+\bar{v}\frac{\partial v'}{\partial y}=-\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial y}-fu'$$

$$\frac{\partial w'}{\partial t}+\bar{u}\frac{\partial w'}{\partial x}+\bar{v}\frac{\partial w'}{\partial y}=-\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z}-\frac{\rho'}{\bar{\rho}}g$$

$$\frac{\partial \rho'}{\partial t}+\bar{u}\frac{\partial \rho'}{\partial x}+\bar{v}\frac{\partial \rho'}{\partial y}+w'\frac{d\bar{\rho}}{dz}=-\bar{\rho}\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)$$

equation of state: $p = \rho R T$

For this set of equations it is now possible to find the wave solutions analytically.

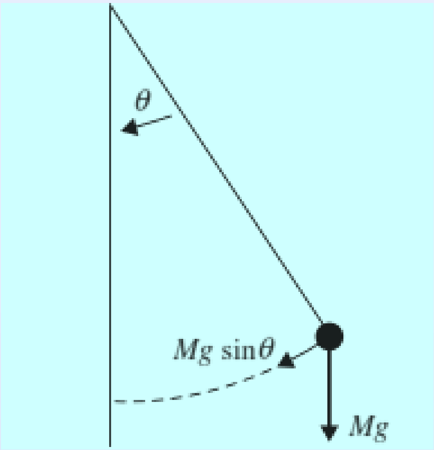
$$\frac{d(\ln T)}{dt}=\frac{R}{c_p}\frac{d(\ln p)}{dt}$$

Thermodynamic equation

Properties of Waves

Wave motions = oscillations in field variables that propagate in space

Familiar example of oscillation: the pendulum



$$m\frac{dV}{dt} = -mg\sin\theta$$ Equation of motion

$$m\frac{d\left(l\dot{\theta}\right)}{dt} = -mg\theta$$ $\sin\theta \approx \theta$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$ where, $\omega^2 = \frac{g}{l}$

The harmonic oscillator equation has the general solution:

$$\theta = \theta_1 \cos\omega t + \theta_2 \sin\omega t = \theta_0 \cos(\omega t - \alpha)$$

Where θ_1, θ_2, θ_0 *and* α *are constants determined by the initial conditions*

$$\theta = \theta_0 \,\mathrm{Re}\left[e^{i(\omega t - \alpha)}\right] = \mathrm{Re}\left[\theta_0 e^{i(\omega t - \alpha)}\right] = \mathrm{Re}\left[\theta_0 e^{-i\alpha} e^{i\omega t}\right]$$

$$\theta = \mathrm{Re}\left[C e^{i\omega t}\right]$$

A sinusoidal wave propagating in the x direction is given by

$$\psi(x,t) = \mathrm{Re}\left[C e^{i(kx - \omega t)}\right] \qquad C = complex\ amplitude = |C| e^{i\phi_c}$$

Phase velocity

How does a point of constant phase move through space?

Thus, for a one-dimensional wave propagating in the *x direction,*

$$\phi(x,t) = kx - \omega t - \alpha = const$$

k = wave number, ω = angular frequency = $2\pi f$ (here f = frequency)

$\phi = kx - \omega t =$ phase, or $\phi = k\left(x - \frac{\omega}{k}t\right)$

For propagating waves the phase is constant for an observer moving at the phase speed

$$c \equiv \frac{\omega}{k}$$

This may be verified by observing that if phase is to remain constant following the motion,

$$\frac{d\phi}{dt} = 0 \qquad \frac{d}{dx}(kx - \omega t - \alpha) = 0 \qquad k\frac{dx}{dt} - \omega = 0$$

$$\text{Phase speed} \quad c = \frac{dx}{dt} = \frac{\omega}{k}$$

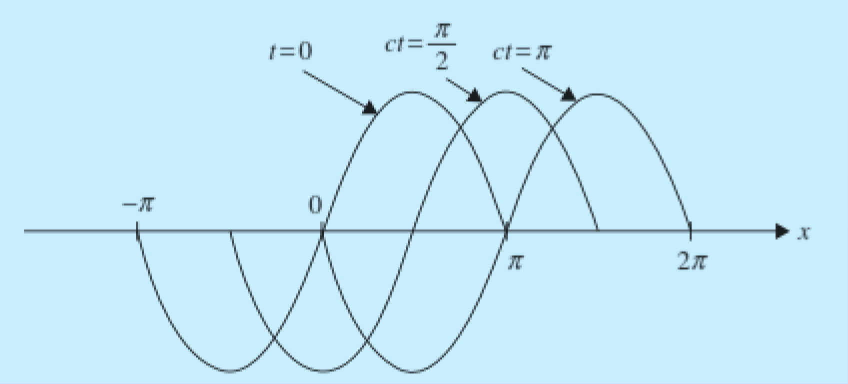
If observer is moving with the wave, then phase is constant.

This gives the change in position x in time t, hence speed, for point maintaining constant phase with respect to wave.

$$\text{for} \quad \omega > 0 \quad \text{and} \quad k > 0 \quad \rightarrow \quad c > 0$$

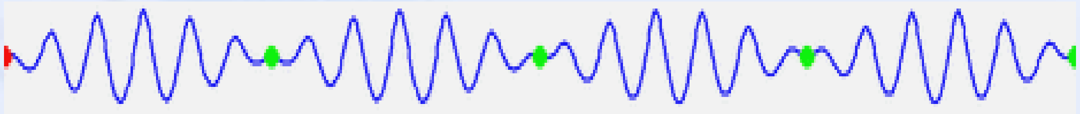
$$\text{if} \quad \alpha = 0 \;, \; \phi = k(x - ct)$$

Wave number is assumed to be unity.



The phase velocity of a wave is the rate at which the phase of the wave propagates in space.

any given phase of the wave (for example, the crest) will appear to travel at the phase velocity.



The red dot moves with the phase velocity, and the green dots propagate with the group velocity.

The phase velocity is twice the group velocity.

The red dot overtakes two green dots when moving from the left to the right of the figure.