Dynamical Meteorology 1

Lecture 8

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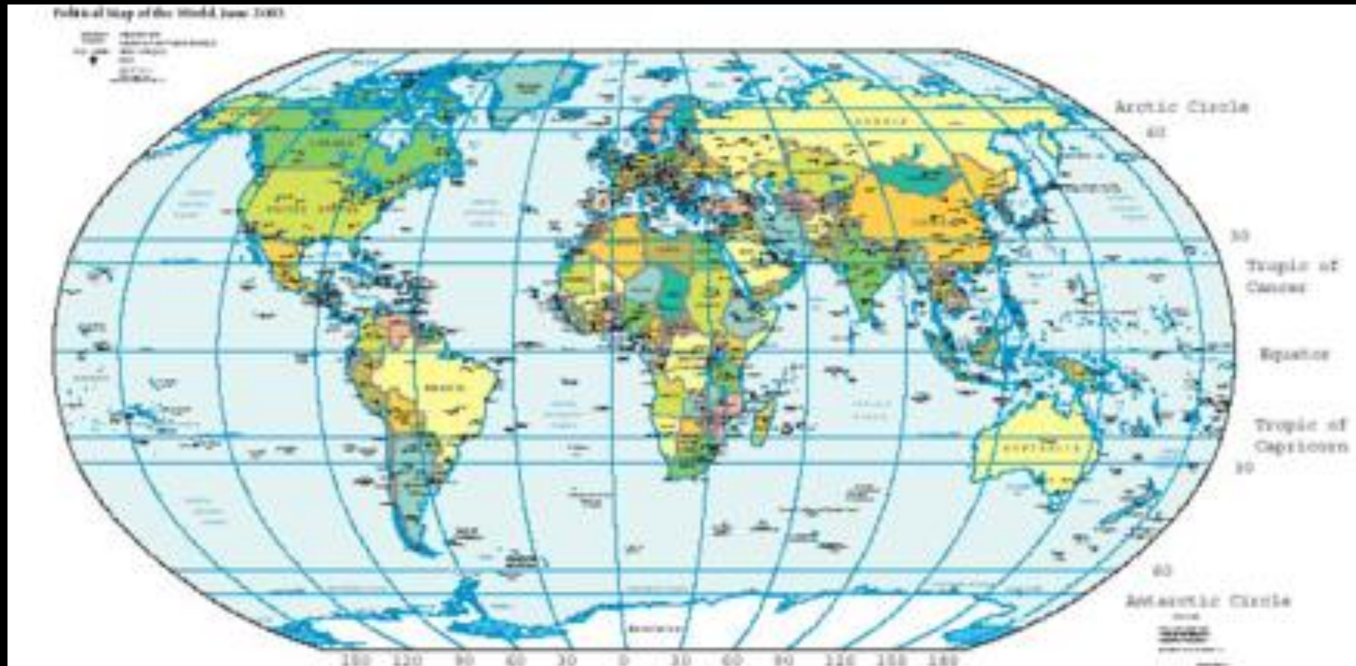
Momentum Equations in Spherical Coordinates

The coordinate axes are (λ, ϕ, z) where λ is longitude, ϕ is latitude, and z is height.

Orientation of Coordinate Axes

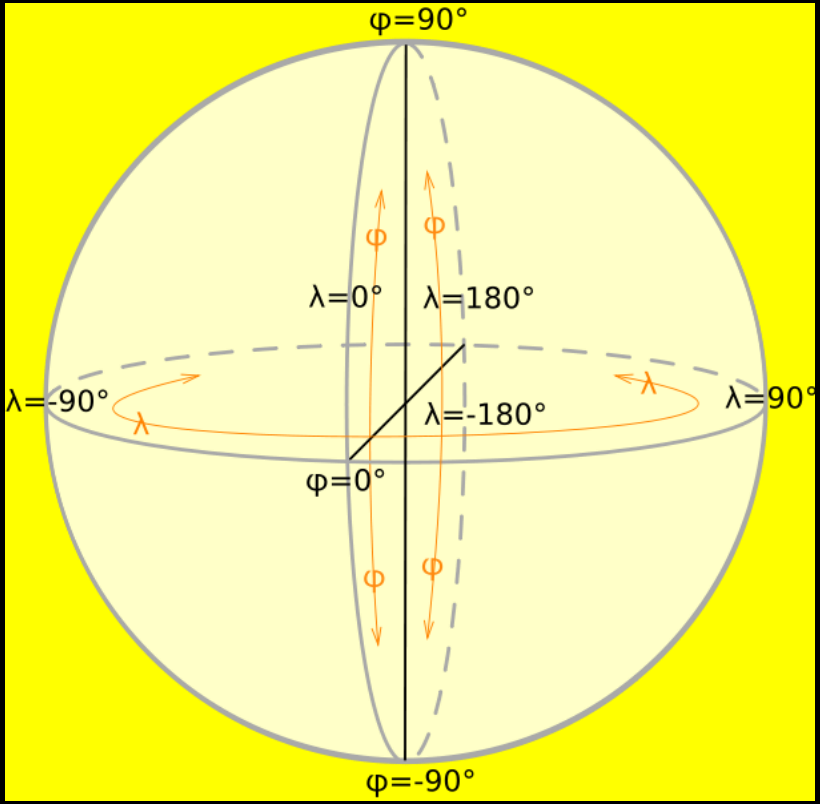
If the unit vectors **i**, **j**, **k** are now taken to be directed eastward, northward, and upward, respectively, the relative velocity become:

$$\vec{V} = \hat{i}u + \hat{j}v + \hat{k}w$$



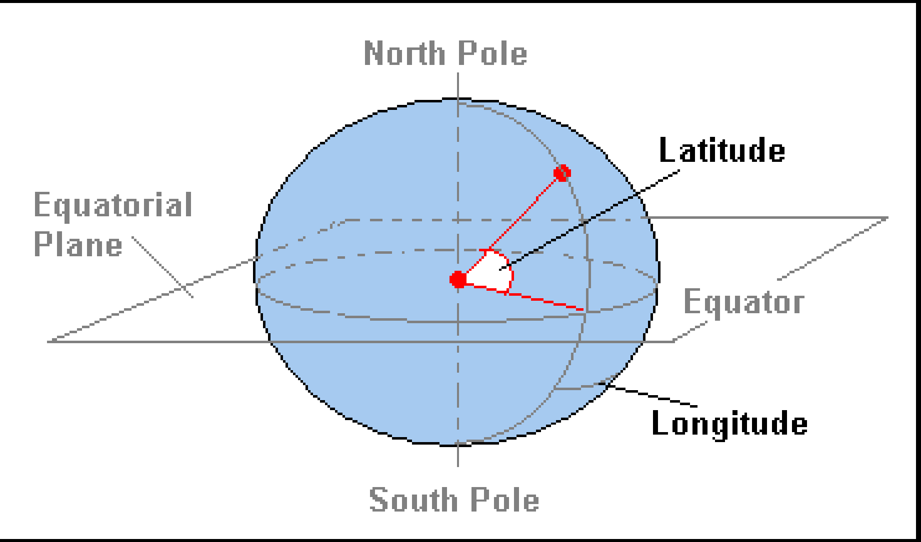
Map of Earth showing lines of latitude (horizontally) and longitude (vertically) latitude is the angle between any point and the equator

a great circle is the equator (latitude=0 degrees), with each pole being 90 degrees north pole 90° N; south pole 90° S).



The anti-meridian of Greenwich is both 180°W and 180°E.

longitude is the angle east or west of an arbitrary point on Eart


North Pole
Latitude
Equatorial
Plane
Equator
Longitude
South Pole

$$dx = R d\lambda = a \cos\phi \, d\lambda$$

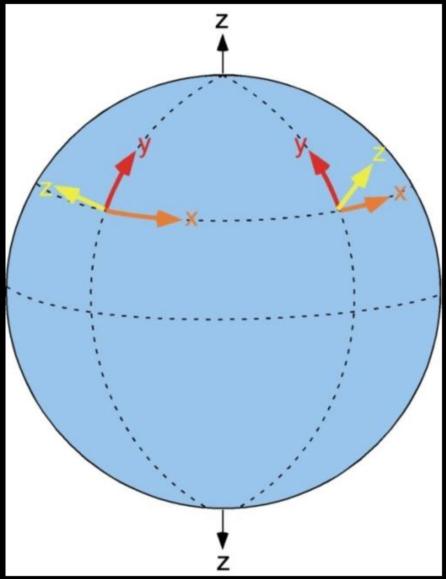
and

Thus the horizontal velocity components are

$$u = \frac{dx}{dt}, \quad v = \frac{dy}{dt}, \quad w = \frac{dr}{dt}$$

The component Equations in Spherical Coordinates

x = positive toward east
y = positive toward north
z = positive toward zenith



$$\frac{dx}{dt} = u$$ wind component toward east

$$\frac{dy}{dt} = v$$ wind component toward north

$$\frac{dz}{dt} = w$$ wind component toward zenith

$$\frac{dx}{dt} = a\cos\varphi\frac{d\lambda}{dt}$$

λ= longitude $$dx = a\cos\varphi d\lambda$$

$$\frac{dy}{dt} = a\frac{d\varphi}{dt}$$

ϕ= latitude $$dy = ad\varphi$$

$$\frac{dz}{dt} = w$$

$$dz = dr$$ r=a+z

Where unit vectors are a function of position on the earth (not Cartesian)

The (x,y,z) coordinates system defined in this way is not a Cartesian coordinates system , because the directions of the unit vectors depend on their position on the earth's surface.

This position dependence of the unit vectors must be taken into account when the acceleration vector is expended into its components on the sphere. Thus, we write:

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

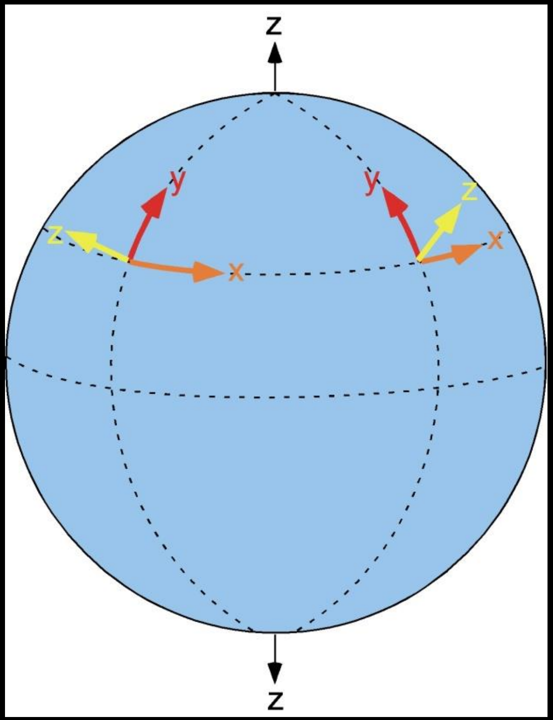
$$\frac{d\vec{V}}{dt} = \hat{i}\frac{du}{dt} + \hat{j}\frac{dv}{dt} + \hat{k}\frac{dw}{dt} + u\frac{d\hat{i}}{dt} + v\frac{d\hat{j}}{dt} + w\frac{d\hat{k}}{dt}$$

The unit vectors are not constan

We need to determine what these are

We first consider $d\hat{i}/dt$

$$\frac{d\hat{i}}{dt} = \cancel{\frac{\partial \hat{i}}{\partial t}} + u\frac{\partial \hat{i}}{\partial x} + \cancel{v\frac{\partial \hat{i}}{\partial y}} + \cancel{w\frac{\partial \hat{i}}{\partial z}}$$



At a point (constant x,y,z) none of the unit vectors change with time so $\frac{\partial \hat{i}}{\partial t} = 0$

As one moves north or south the i direction experiences no change so $\frac{\partial \hat{i}}{\partial y} = 0$

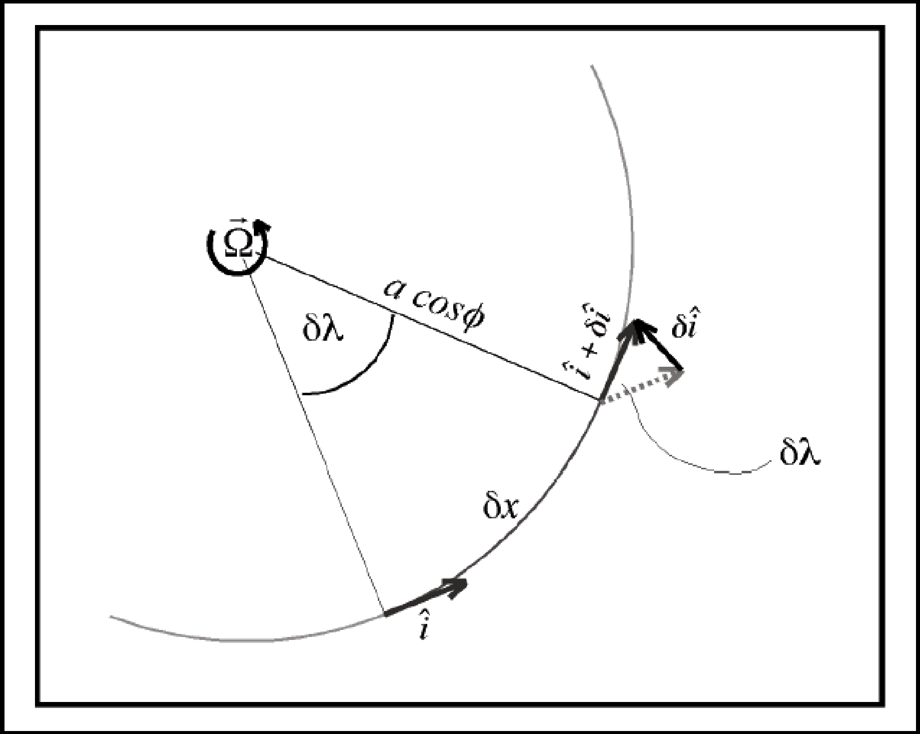
As one moves up or down the i direction experiences no change so $\frac{\partial \hat{i}}{\partial z} = 0$

So: $$\frac{d\hat{i}}{dt} = u\frac{\partial \hat{i}}{\partial x}$$

i is a function only of *x*

$$\frac{d\hat{i}}{dt} = u\frac{\partial\hat{i}}{\partial x}$$

From figure on right looking down at north pole at latitude ϕ:



$$\left|\delta\hat{i}\right| = \left|\hat{i}\right|\delta\lambda$$

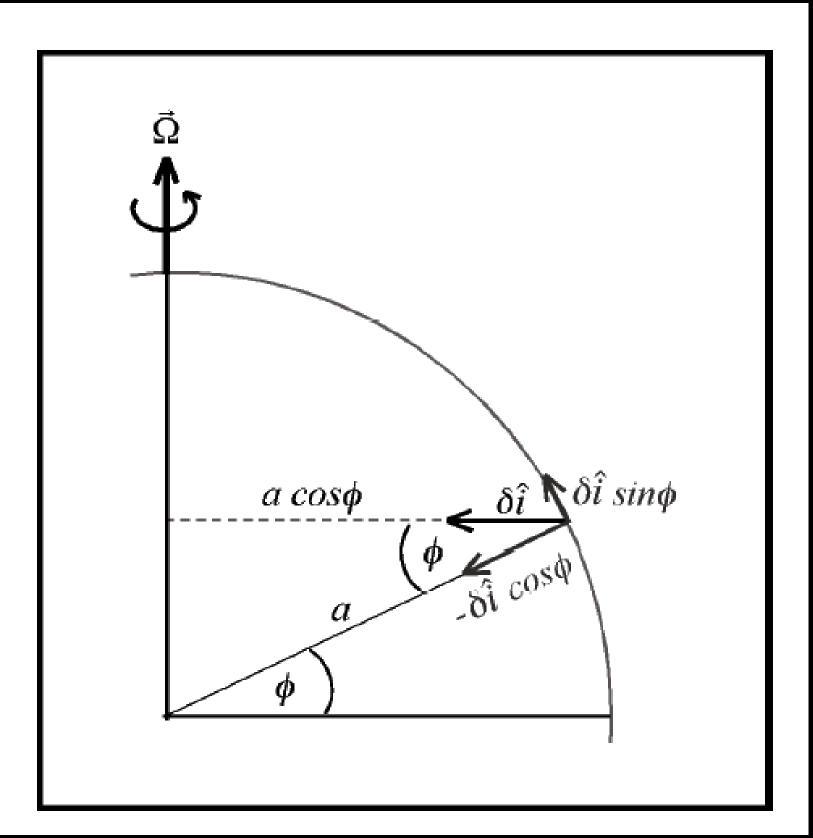
$$\delta x = a\cos\varphi\,\delta\lambda$$

$$\left|\frac{\delta\hat{i}}{\delta x}\right| = \frac{1}{a\cos\varphi}$$

This gives us the magnitude, but not the direction.

Note that is $\partial\hat{i}$ *pointed toward the center of the earth at the original point*

We see that $\delta\hat{i}$ has components in the $\hat{j}$ and $-\hat{k}$ directions



The unit vector describing the direction of $\delta\hat{i}$ has two components:

So: $$u\frac{d\hat{i}}{dt} = u\frac{\partial\hat{i}}{\partial x} = \frac{u(\hat{j}\sin\varphi - \hat{k}\cos\varphi)}{a\cos\varphi} = \hat{j}\frac{u\tan\varphi}{a} - \hat{k}\frac{u}{a}$$

REMEMBER

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

Since the unit vectors are not constan

$$\frac{d\vec{V}}{dt} = \hat{i}\frac{du}{dt} + \hat{j}\frac{dv}{dt} + \hat{k}\frac{dw}{dt} + u\left(\hat{j}\frac{u\tan\varphi}{a} - \hat{k}\frac{u}{a}\right) + v\frac{d\hat{j}}{dt} + w\frac{d\hat{k}}{dt}$$

We still need to determine what these are

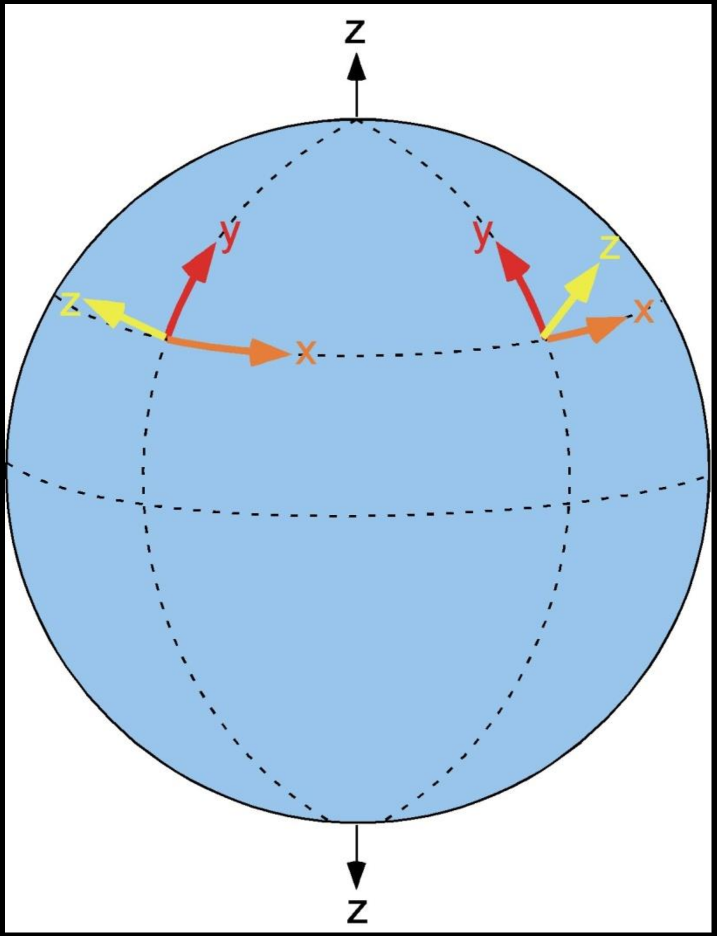
Let's do $\frac{d\hat{j}}{dt}$ next

$$\frac{d\hat{j}}{dt} = \frac{\partial\hat{j}}{\partial t} + u\frac{\partial\hat{j}}{\partial x} + v\frac{\partial\hat{j}}{\partial y} + w\frac{\partial\hat{j}}{\partial z}$$

$$\frac{d\hat{j}}{dt} = \cancel{\frac{\partial \hat{j}}{\partial t}} + u\frac{\partial \hat{j}}{\partial x} + v\frac{\partial \hat{j}}{\partial y} + w\cancel{\frac{\partial \hat{j}}{\partial z}}$$

$\hat{j}$ does not change with time or elevation, but does change in the x and y directions

$$\frac{d\hat{j}}{dt} = u\frac{\partial \hat{j}}{\partial x} + v\frac{\partial \hat{j}}{\partial y}$$



Lets first figure out

Look at light gray triangle in (a):

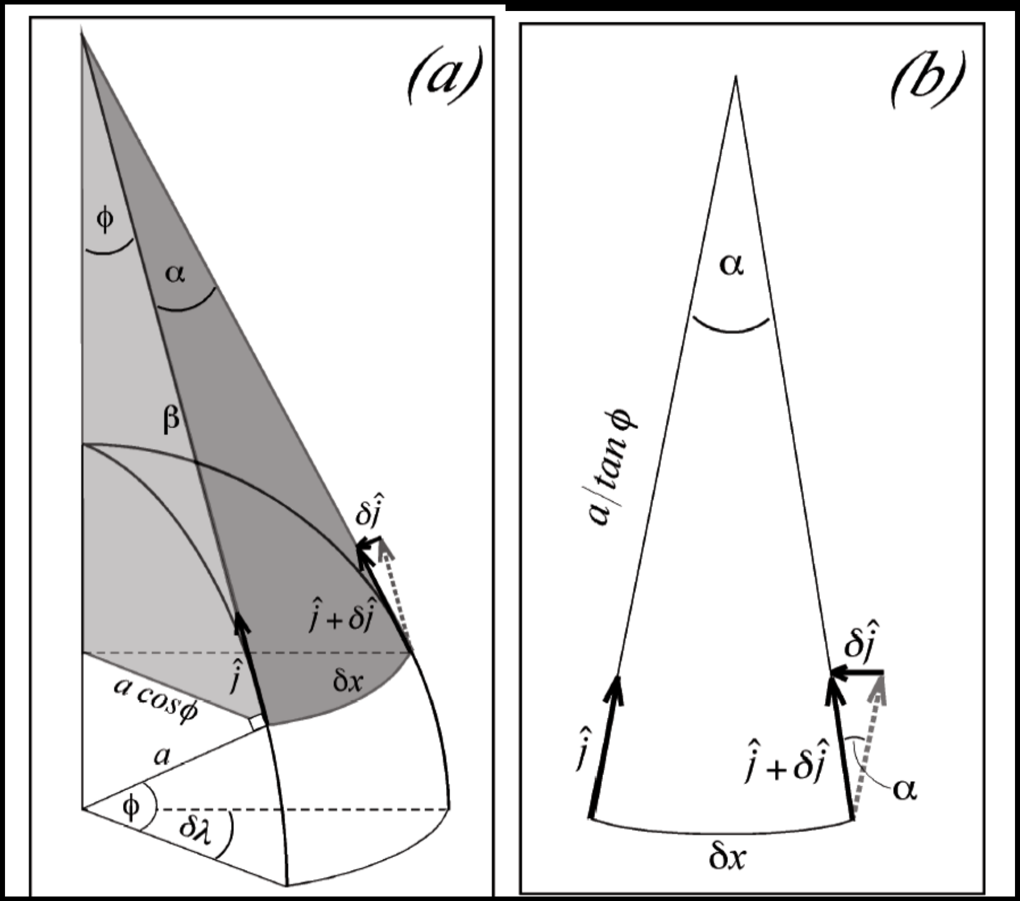
$$a\cos\varphi = \beta\sin\varphi$$

or $$\beta = \frac{a}{\tan\varphi}$$

Dark gray triangle is shown in a different view in (b)

$$\delta x = \frac{a}{\tan\varphi}\delta\alpha$$

$$\delta\hat{j} = \hat{j}\,\delta\alpha$$



So: $$\left|\frac{\delta\hat{j}}{\delta x}\right| = \frac{\tan\varphi}{a}$$

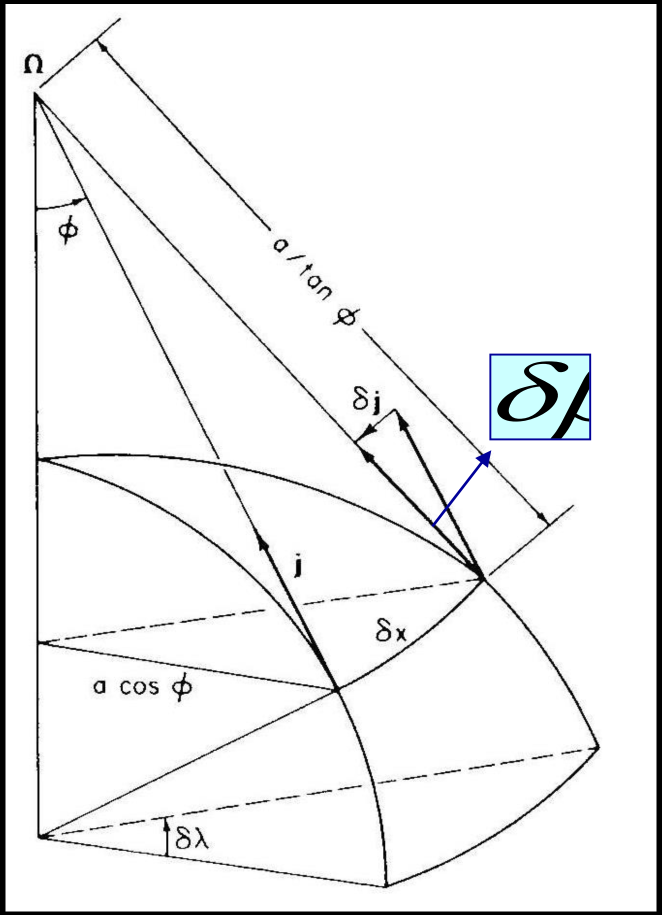
Direction of is the -x direction

Thus, with the aid of above this fig. we see that for eastward motion:

$$\left|\delta\hat{j}\right| = \delta\beta$$

$$= \delta x / y$$

$$= \delta x /(a / tg\phi)$$



Since the vector $\partial \hat{j} / \partial x$ is directed in the negative x direction,

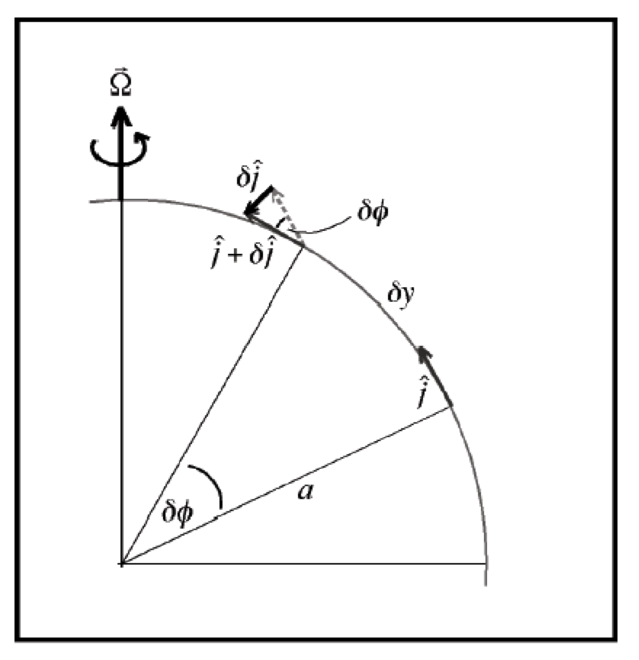
$$\lim_{\delta x \to 0} \frac{\delta \hat{j}}{\delta x} = \frac{\partial \hat{j}}{\partial x} = -\frac{tg\phi}{a}\hat{i}$$

From Fig. it is clear that for northward motion

$$\left|\delta \hat{j}\right| = \delta \phi$$

$$\delta y = a\delta \phi$$

$$\lim_{\delta y \to 0} \frac{\delta \hat{j}}{\delta y} = \frac{\partial \hat{j}}{\partial y} = -\frac{1}{a}\hat{k}$$



$$\frac{d\hat{j}}{dt} = u\frac{\partial \hat{j}}{\partial x} + v\frac{\partial \hat{j}}{\partial y}$$

$$\frac{d\hat{j}}{dt} = -\frac{utg\phi}{a}\hat{i} - \frac{v}{a}\hat{k}$$

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$\frac{d\vec{V}_i}{dt} = \hat{i}\frac{du}{dt} + \hat{j}\frac{dv}{dt} + \hat{k}\frac{dw}{dt} + u\left(\hat{j}\frac{u\tan\varphi}{a} - \hat{k}\frac{u}{a}\right) + v\left(-\hat{i}\frac{u\tan\varphi}{a} - \hat{k}\frac{v}{a}\right) + w\frac{d\hat{k}}{dt}$$

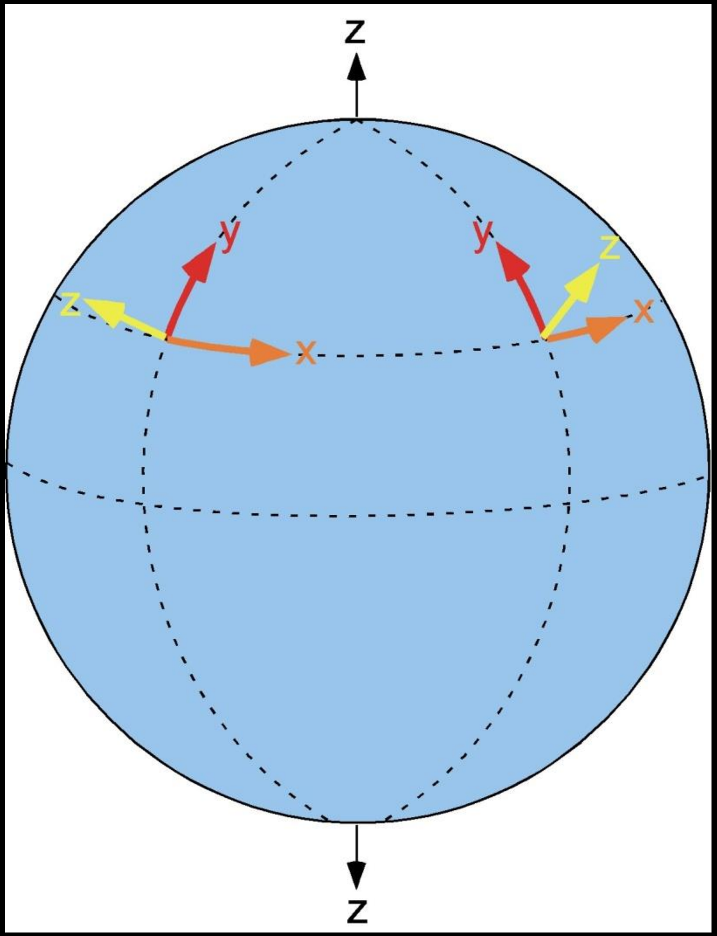
Let's do $\frac{d\hat{k}}{dt}$ next

$$\frac{d\hat{k}}{dt} = \frac{\partial\hat{k}}{\partial t} + u\frac{\partial\hat{k}}{\partial x} + v\frac{\partial\hat{k}}{\partial y} + w\frac{\partial\hat{k}}{\partial z}$$

$$\frac{d\hat{k}}{dt} = \cancel{\frac{\partial \hat{k}}{\partial t}} + u\frac{\partial \hat{k}}{\partial x} + v\frac{\partial \hat{k}}{\partial y} + \cancel{w\frac{\partial \hat{k}}{\partial z}}$$

$\hat{k}$ does not change with time or elevation, but does change in the x and y directions

$$\frac{d\hat{k}}{dt} = u\frac{\partial \hat{k}}{\partial x} + v\frac{\partial \hat{k}}{\partial y}$$



$$\frac{d\hat{k}}{dt} = u\frac{\partial \hat{k}}{\partial x} + v\frac{\partial \hat{k}}{\partial y}$$

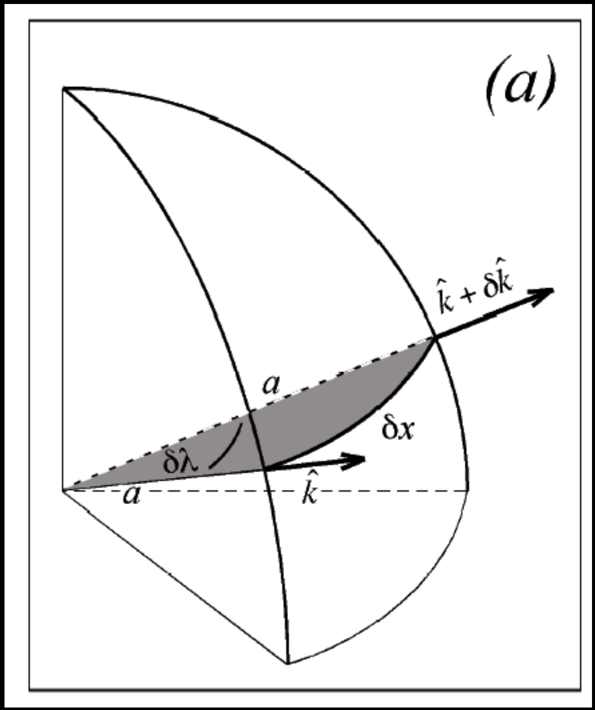
Let's do $\frac{\partial \hat{k}}{\partial x}$ first

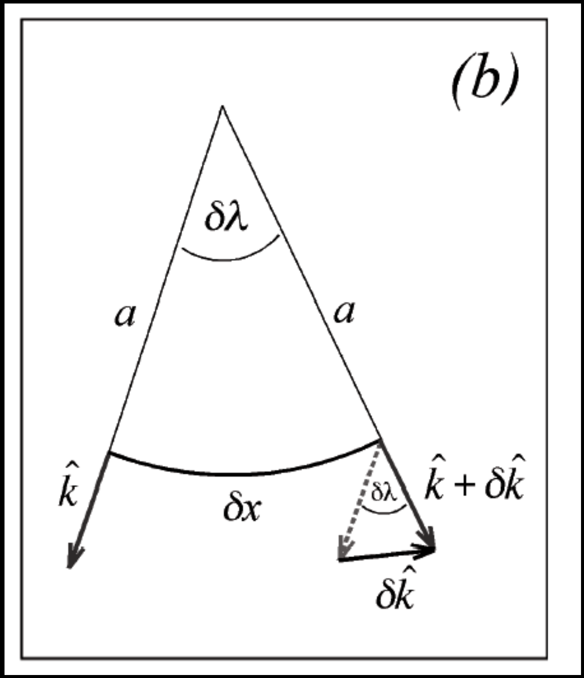
$$\delta x = a\delta\lambda$$

$$\left|\delta\hat{k}\right| = \left|\hat{k}\,\delta\lambda\right| = \delta\lambda$$

Direction of $\delta\hat{k}$ is the positive $\hat{i}$ direction

$$\frac{\partial \hat{k}}{\partial x} = \frac{1}{a}\hat{i} \qquad u\frac{\partial \hat{k}}{\partial x} = \hat{i}\,\frac{u}{a}$$



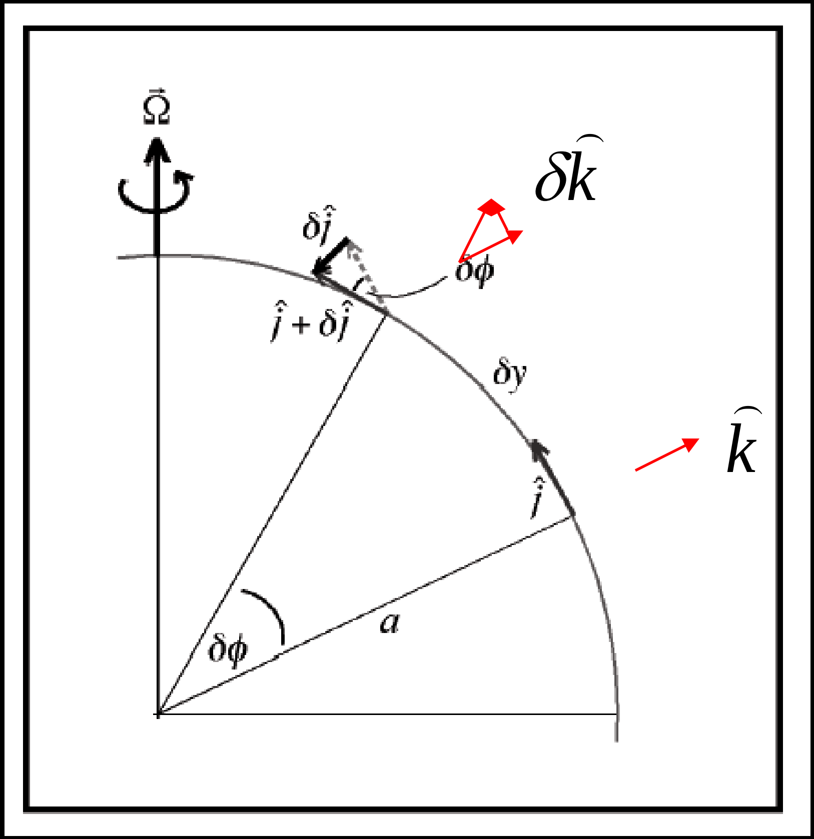


$$\frac{d\hat{k}}{dt} = u\frac{\partial \hat{k}}{\partial x} + v\frac{\partial \hat{k}}{\partial y}$$

Let's do $\frac{\partial \hat{k}}{\partial y}$ next

$$\delta y = a\delta\varphi$$

$$\left|\delta\hat{k}\right| = \left|\hat{k}\,\delta\varphi\right| = \delta\varphi$$



Direction of $\delta\hat{k}$ is the positive $\hat{i}$ direction

$$\frac{\partial \hat{k}}{\partial y} = \frac{1}{a}\hat{j}$$

$$\frac{d\hat{k}}{dt} = \frac{u}{a}\hat{i} + \frac{v}{a}\hat{j}$$

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$\frac{d\vec{V}}{dt} = \hat{i}\frac{du}{dt} + \hat{j}\frac{dv}{dt} + \hat{k}\frac{dw}{dt} + u\left(\hat{j}\frac{u\tan\varphi}{a} - \hat{k}\frac{u}{a}\right) + v\left(-\hat{i}\frac{u\tan\varphi}{a} - \hat{k}\frac{v}{a}\right) + w\left(\frac{u}{a}\hat{i} + \frac{v}{a}\hat{j}\right)$$

or

$$\frac{d\vec{V}}{dt} = \hat{i}\left(\frac{du}{dt} - \frac{uv\tan\varphi}{a} + \frac{uw}{a}\right) + \hat{j}\left(\frac{dv}{dt} + \frac{u^2\tan\varphi}{a} + \frac{vw}{a}\right) + \hat{k}\left(\frac{dw}{dt} - \frac{u^2 - v^2}{a}\right)$$

A Complication of Spherical Coordinates

Newton's second law in an inertial coordinate system

$$m\frac{d\vec{V}_i}{dt} = \sum Forces\ acting\ on\ an\ object$$

Newton's second law in a spherical coordinate system

$$m\left[\frac{d\vec{V}}{dt} + \hat{i}\left(-\frac{uv\tan\varphi}{a} + \frac{uw}{a}\right) + \hat{j}\left(\frac{u^2\tan\varphi}{a} + \frac{vw}{a}\right) + \hat{k}\left(-\frac{u^2 - v^2}{a}\right)\right] = \sum Forces$$

Now let's look at the right side of the equation!

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

Vector momentum equation in rotating coordinates

Total derivative

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = \ldots$$

$$\frac{dv}{dt} + \frac{u^2\tan\varphi}{a} + \frac{vw}{a} = \ldots$$

$$\frac{dw}{dt} - \frac{u^2+v^2}{a} = \ldots$$

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

Coriolis acceleration

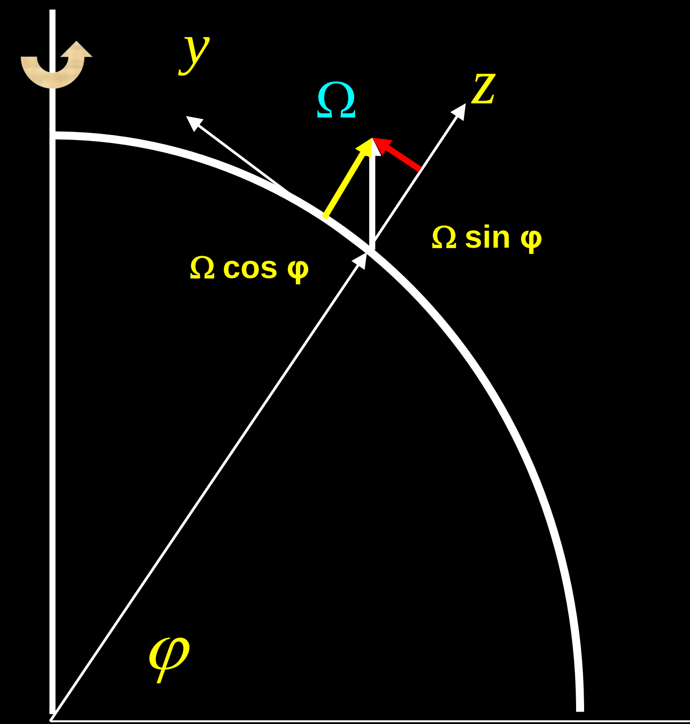
$$-2\vec{\Omega}\times\vec{V} = -2\Omega\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \cos\varphi & \sin\varphi \\ u & v & w \end{vmatrix} = \left(2\Omega v\sin\varphi - 2\Omega w\cos\varphi\right)\hat{i}$$

$$-2\Omega u\sin\varphi\,\hat{j} + 2\Omega\cos\varphi\,\hat{k}$$

$$\frac{du}{dt} - \frac{uv\tan\varphi}{a} + \frac{uw}{a} = 2\Omega v\sin\varphi - 2\Omega w\cos\varphi + \ldots$$

$$\frac{dv}{dt} + \frac{u^2\tan\varphi}{a} + \frac{vw}{a} = -2\Omega u\sin\varphi + \ldots$$

$$\frac{dw}{dt} - \frac{u^2+v^2}{a} = 2\Omega u\cos\varphi + \ldots$$



$$\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

Pressure gradient term

$$\nabla p = \frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}$$

$$\frac{du}{dt} - \frac{uv\tan\varphi}{a} + \frac{uw}{a} = 2\Omega v\sin\varphi - 2\Omega w\cos\varphi - \frac{1}{\rho}\frac{\partial p}{\partial x} + \ldots$$

$$\frac{dv}{dt} + \frac{u^2\tan\varphi}{a} + \frac{vw}{a} = -2\Omega u\sin\varphi - \frac{1}{\rho}\frac{\partial p}{\partial y} + \ldots$$

$$\frac{dw}{dt} - \frac{u^2 + v^2}{a} = 2\Omega u\cos\varphi - \frac{1}{\rho}\frac{\partial p}{\partial z} + \ldots$$

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

Gravity

$\vec{g} = -g\hat{k}$ g is a positive scalar = 9.8 m s^{-2} at earth's surface

$$\frac{du}{dt} - \frac{uv\tan\varphi}{a} + \frac{uw}{a} = 2\Omega v\sin\varphi - 2\Omega w\cos\varphi - \frac{1}{\rho}\frac{\partial p}{\partial x} + \ldots$$

$$\frac{dv}{dt} + \frac{u^2\tan\varphi}{a} + \frac{vw}{a} = -2\Omega u\sin\varphi - \frac{1}{\rho}\frac{\partial p}{\partial y} + \ldots$$

$$\frac{dw}{dt} - \frac{u^2 + v^2}{a} = 2\Omega u\cos\varphi - \frac{1}{\rho}\frac{\partial p}{\partial z} - g + \ldots$$

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

Friction

$$\vec{F}_r = F_{rx}\hat{i} + F_{ry}\hat{j} + F_{rz}\hat{k}$$

$$\frac{du}{dt} - \frac{uv\tan\varphi}{a} + \frac{uw}{a} = 2\Omega v\sin\varphi - 2\Omega w\cos\varphi - \frac{1}{\rho}\frac{\partial p}{\partial x} + F_{rx}$$

$$\frac{dv}{dt} + \frac{u^2\tan\varphi}{a} + \frac{vw}{a} = -2\Omega u\sin\varphi - \frac{1}{\rho}\frac{\partial p}{\partial y} + F_{ry}$$

$$\frac{dw}{dt} - \frac{u^2 + v^2}{a} = 2\Omega u\cos\varphi - \frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_{rz}$$

Momentum Equations in Spherical Coordinates

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - 2\Omega w\cos\phi + F_{rx}$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi + F_{ry}$$

$$\frac{dw}{dt} - \frac{u^2+v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + 2\Omega u\cos\phi - g + F_{rz}$$

The terms proportional to *1/a* on the left hand sides are called the curvature terms; they arise owing to the curvature of the earth.

Because they are nonlinear (that is, they are quadratic in the dependent variables) they are difficult to handle in theoretical analyses.

Any Questions?