Differential Equations

Lecture 7

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Verify that the following equation are homogeneous and solve them.

$$x^2 y' - 3xy - 2y^2 = 0$$

$$\frac{dy}{dx} = \frac{3y}{x} + \frac{2y^2}{x^2} = \frac{3xy + 2y^2}{x^2}$$

$$f(x, y) = \frac{3xy + 2y^2}{x^2}$$

$$f(tx, ty) = \frac{t^2(3xy + 2y^2)}{t^2 x^2} = t^0 f(x, y)$$

$$f(x,y)=\frac{3y/x+2y^2/x^2}{x^2/x^2}$$

$$\frac{dy}{dx}=\frac{3y/x+2y^2/x^2}{x^2/x^2} \qquad \frac{y}{x}=z$$

$$\frac{dy}{dx}=3z+2z^2 \qquad y=xz \rightarrow \frac{dy}{dx}=z+x\frac{dz}{dx}$$

$$z+x\frac{dz}{dx}=3z+2z^2 \qquad x\frac{dz}{dx}=2z+2z^2$$

$$2\int\frac{dx}{x}=\int\frac{dz}{z+z^2}=\int\frac{dz}{z}-\int\frac{dz}{1+z}$$

$$2\ln x+\ln c=\ln z-\ln(1+z)$$

$$cx^2(1+y/x)=y/x$$

$$y=cx^2(x+y)$$

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Show that the substitution *z=ax+by+c* changes

$$y' = f(ax+by+c)$$

Into an equation with separable variables, and apply this method to solve the following equations:

$$y' = f(ax+by+c) = f(z)$$

$$\frac{1}{b}\left(\frac{dz}{dx}-a\right)=\frac{dy}{dx}=f(z)$$

$$z = ax+by+c$$

$$\frac{dz}{bf(z)+a}=dx$$

$$\frac{dz}{dx}=a+b\frac{dy}{dx}$$

$$\frac{dz}{g(z)}=dx$$

$$a)\ \ y' = (x+y)^2$$

$$z = x + y \rightarrow z' = 1 + y' \rightarrow y' = z' - 1$$

$$z' - 1 = z^2 \rightarrow z' = z^2 + 1$$

$$\frac{dz}{dx} = z^2 + 1 \rightarrow \frac{dz}{z^2+1} = dx$$

$$\int \frac{dz}{z^2+1} = \int dx \qquad \tan^{-1} z = x + c$$

$$\tan^{-1}(x+y) = x + c \qquad x + y = \tan(x+c)$$

$$M(x,y)dx + N(x,y)dy = 0 \quad *$$

$$\mu(M(x,y)dx + N(x,y)dy) = 0$$

$$f(x,y) = c \quad \text{g. s.}*$$

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \quad **$$

$$*,** \to \frac{dy}{dx} = -\frac{M}{N} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

$$\frac{\partial f/\partial x}{M} = \frac{\partial f/\partial y}{N} = \mu \qquad \frac{\partial f}{\partial x} = \mu M \;, \; \frac{\partial f}{\partial y} = \mu N$$

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

$$\frac{1}{\mu}\left(N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y}\right) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$$

Much more difficult problem respect to *

$$Case\,1)\quad \mu(x),\quad \frac{\partial \mu}{\partial y} = 0 \qquad Case\,2)\quad \mu(y),\quad \frac{\partial \mu}{\partial x} = 0$$

$$g(x) = \frac{\partial M/\partial y - \partial N/\partial x}{N} \qquad h(y) = \frac{\partial M/\partial y - \partial N/\partial x}{-M}$$

$$\mu = e^{\int g(x)dx} \qquad \mu = e^{\int h(y)dy}$$

Case 3 μ(z)

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$$\mu\frac{\partial M}{\partial y}+M\frac{\partial \mu}{\partial y}=\mu\frac{\partial N}{\partial x}+N\frac{\partial \mu}{\partial x}$$

$$\frac{\partial \mu}{\partial y}=\frac{\partial \mu}{\partial z}\frac{\partial z}{\partial y}$$

$$z=xy$$

$$\frac{\partial \mu}{\partial y}=\frac{\partial \mu}{\partial z}x$$

$$\frac{\partial \mu}{\partial x}=\frac{\partial \mu}{\partial z}\frac{\partial z}{\partial x}$$

$$\frac{\partial \mu}{\partial x}=\frac{\partial \mu}{\partial z}y$$

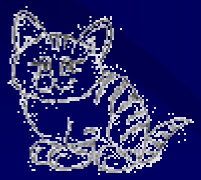
$$\mu(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})=Ny\frac{d\mu}{dz}-Mx\frac{d\mu}{dz}$$

$$\mu(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})=(Ny-Mx)\frac{d\mu}{dz}$$

$$\frac{\partial M / \partial y - \partial N / \partial x}{Ny - Mx} = \frac{1}{\mu}\frac{d\mu}{dz}$$

$$g(z)dz = \frac{d\mu}{\mu}$$

$$\mu(z) = e^{\int g(z)dz}$$



Solve each of the following equations by finding an integrating factor

$$xdy + ydx + 3x^3y^4dy = 0$$

$$ydx + (3x^3y^4 + x)dy = 0$$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 9x^2y^4 + 1$$

$$\frac{\partial M/\partial y - \partial N/\partial x}{Ny - Mx} = \frac{1-(9x^2y^4+1)}{(3x^3y^4+x)y - xy} = -\frac{3}{xy}$$

$$z = xy$$

$$\mu(z) = e^{\int g(z)dz} = e^{\int(-3/z)dz} = z^{-3} = \frac{1}{z^3}$$

$$\mu = \frac{1}{x^3 y^3}$$

$$\frac{1}{x^3 y^3}(ydx + (3x^3 y^4 + x)dy) = 0$$

$$\frac{1}{x^3 y^2}dx + (3y + \frac{1}{x^2 y^3})dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -\frac{2}{y^3 x^3}$$

$$\frac{\partial f(x,y)}{\partial x} = M = \frac{1}{x^3 y^2}$$

$$f(x,y) = \int \frac{dx}{x^3 y^2} + g(x) = -\frac{1}{2x^2 y^2} + g(y)$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{4yx^2}{4y^4 x^4} + g'(y) = 3y + \frac{1}{x^2 y^3}$$

$$g'(y) = 3y \rightarrow g(y) = \frac{3}{2} y^2$$

$$f(x,y) = \frac{3}{2} y^2 - \frac{1}{2x^2 y^2} \qquad \frac{3}{2} y^2 - \frac{1}{2x^2 y^2} = c$$

$$\mu(x), \quad \frac{\partial \mu}{\partial y} = 0$$

$$g(x) = \frac{\partial M / \partial y - \partial N / \partial x}{N} = \frac{9x^2 y^4}{3x^3 y^4 + x}$$

$$\mu(y), \quad \frac{\partial \mu}{\partial x} = 0$$

$$h(y) = \frac{\partial M / \partial y - \partial N / \partial x}{-M} = 9x^2 y^3$$

Under what circumstances will equation *M(x,y)dx+N(x,y)dy=0* have an integrating factor that is a function of the sum *z=x+y*?

$$\frac{\partial(\mu M)}{\partial y}=\frac{\partial(\mu N)}{\partial x}$$

$$\mu\frac{\partial M}{\partial y}+M\frac{\partial \mu}{\partial y}=\mu\frac{\partial N}{\partial x}+N\frac{\partial \mu}{\partial x}$$

$$\frac{\partial \mu}{\partial y}=\frac{\partial \mu}{\partial z}\frac{\partial z}{\partial y}\quad\rightarrow\frac{\partial \mu}{\partial y}=\frac{\partial \mu}{\partial z}\times 1$$

$$\frac{\partial \mu}{\partial x}=\frac{\partial \mu}{\partial z}\frac{\partial z}{\partial x}\quad\rightarrow\frac{\partial \mu}{\partial x}=\frac{\partial \mu}{\partial z}\times 1$$

$$\mu \frac{\partial M}{\partial y} + M \frac{\partial \mu}{\partial z} = \mu \frac{\partial N}{\partial x} + N \frac{\partial \mu}{\partial z}$$

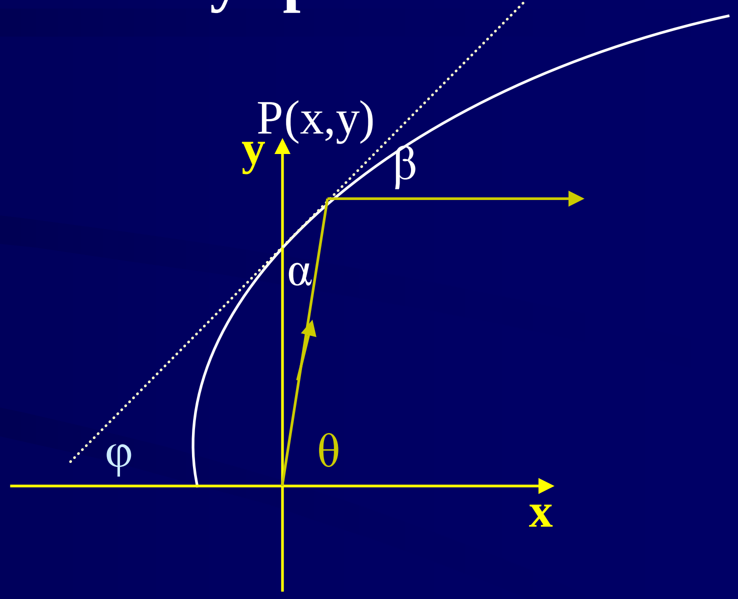
$$\mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = (N - M) \frac{\partial \mu}{\partial z}$$

$$\frac{d\mu}{\mu} = \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N - M} dz = g(z)dz$$

$$\mu = e^{\int g(z)dz}$$

Example

Find the shape of a curved mirror that light from a source at the origin will be reflected in a beam of rays parallel to the *x* axis.



$$\alpha = \beta \qquad \varphi = \beta$$

$$\theta = \alpha + \varphi = 2\beta$$

$$tg\theta = \frac{y}{x} \qquad tg\varphi = \frac{dy}{dx}$$

$$tg\theta = tg2\beta = \frac{2tg\beta}{1 - tg^2\beta} \qquad \frac{y}{x} = \frac{2dy/dx}{1-(dy/dx)^2}$$

$$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$$

Solving this quadratic for *dy/dx* gives:

$$\frac{dy}{dx} = \frac{-x \pm \sqrt{x^2 + y^2}}{y}$$

$$ydy + xdx = \pm\sqrt{x^2 + y^2}\,dx$$

$$\frac{ydy + xdx}{\sqrt{x^2 + y^2}} = \pm dx$$

$$d\sqrt{x^2+y^2} = \pm dx$$

$$\sqrt{x^2+y^2} = \pm x + c$$

$$x^2+y^2 = x^2 \pm 2c\,x + c^2$$

$$y^2 = \pm 2cx + c^2$$

which is the equation of the family of all parabolas with focus at the origin and axis the x axis.

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Solve each of the following equations:

$$d)\ (y+x)dy=(y-x)dx$$

$$ydy+xdy=ydx-xdx \qquad xdx+ydy=ydx-xdy$$

$$\frac{xdx+ydy}{x^2+y^2}=\frac{ydx-xdy}{x^2+y^2}$$

$$\frac{1/2d(x^2+y^2)}{x^2+y^2}=d(\tan^{-1}\frac{y}{x})$$

$$\int\frac{1/2d(x^2+y^2)}{x^2+y^2}=\int d(\tan^{-1}\frac{y}{x})$$

$$\ln\sqrt{x^2+y^2}=\tan^{-1}\frac{y}{x}+c$$

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$$\frac{dx}{dt} = k(x_1 - x)x \qquad \frac{dx}{x(x_1 - x)} = kdt$$

$$\frac{1}{x(x_1 - x)} = \frac{A}{x} + \frac{B}{x_1 - x} \qquad 1 = A(x_1 - x) + B(x)$$

$$Ax_1 = 1 \rightarrow A = \frac{1}{x_1}$$

$$(B - A)x = 0 \rightarrow B - A = 0 \Rightarrow B = A = \frac{1}{x_1}$$

$$\frac{1}{x(x_1 - x)} = \frac{1/x_1}{x} + \frac{1/x_1}{x_1 - x} = \frac{1}{x_1 x} + \frac{1}{x_1(x_1 - x)}$$

$$\int_0^x \frac{dx}{x_1 x} + \int_0^x \frac{dx}{x_1(x_1 - x)} = \int_0^t k\,dt$$

$$\frac{1}{x_1}\ln x - \frac{1}{x_1}\ln(x_1 - x) + c = kt$$

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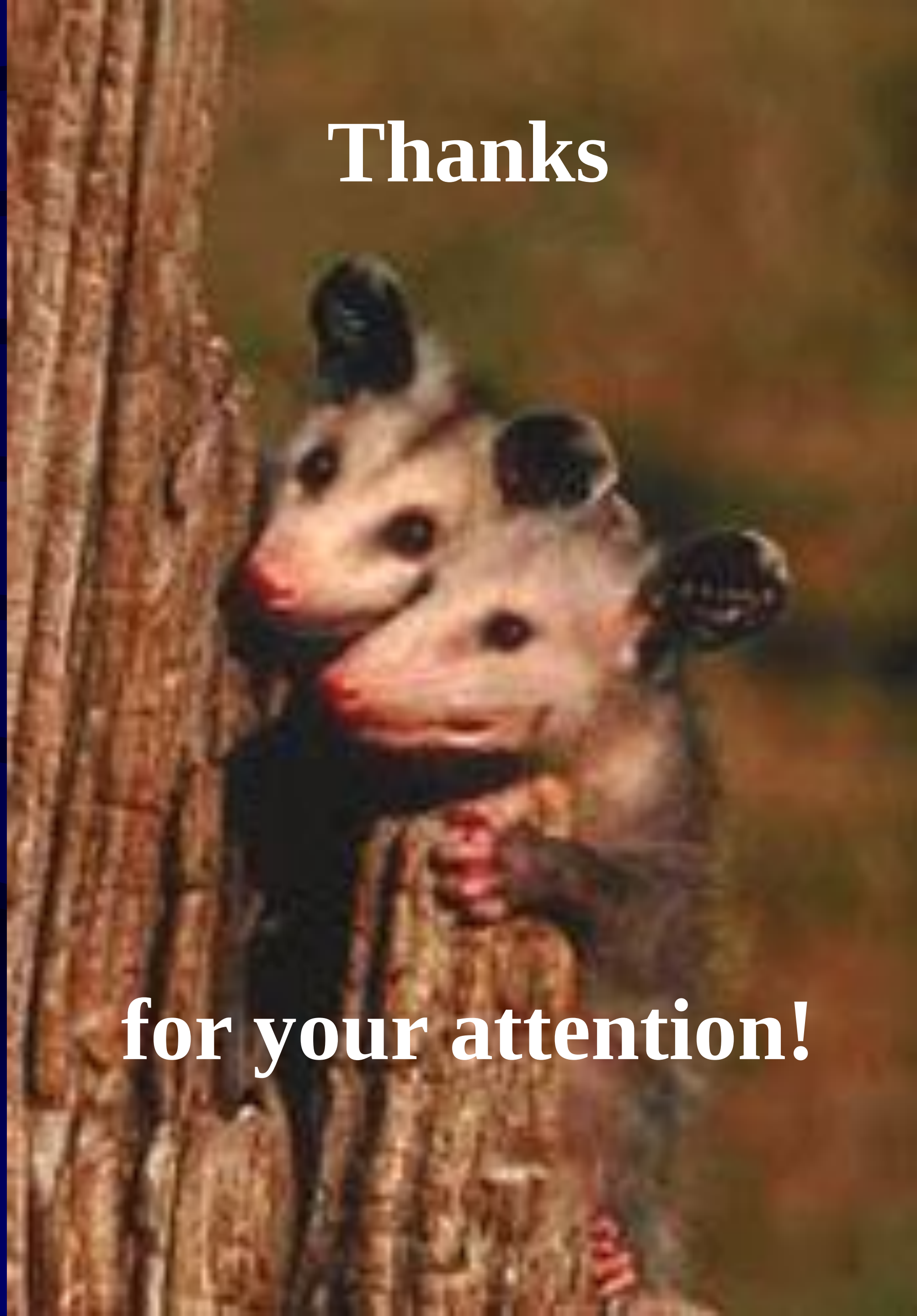
$$dP = -k\rho dh \qquad PV = c_1$$

$$\rho = \frac{m}{V} = \frac{mP}{c_1} \qquad dP = -k\left(\frac{mP}{c_1}\right)dh$$

$$\frac{dP}{P} = -c\,dh \qquad \int_{P_0}^{P} \frac{dP}{P} = -c\int_0^h dh$$

$$\ln\frac{P}{P_0} = -ch \qquad P = P_0 e^{-ch}$$


Thanks
for your attention!