معادلات دیفرانسیل

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Exact Differential

Definition (Exact Differential) **For a function of two variables** $z = f(x,y)$ **if** x **and** y **are given increments** Δx **and** Δy**, then the corresponding increment of z is**

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$

The differentials ***dx*** **and** ***dy*** **are independent variables; that is, they can be given any values. Then the** ***differential dz*** **also called the** ***total differential*****, is defined by**

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Exact Differential Equations

If we have a family of curves $f(x,y)=c$, then its differential equation can be written in the form:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

For example, the family $x^2y^3=c$ has:

$$2xy^3dx + 3x^2y^2dy = 0$$

Suppose we turn this situation around, and begin with the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

A first –order differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

If there happens to exist a function ***f(x,y)*** **such that:**

$$\frac{\partial f}{\partial x} = M \qquad \text{and} \qquad \frac{\partial f}{\partial y} = N$$

an exact differential equation if the differential form *M*(*x*, *y*) *dx* + *N*(*x*, *y*) *dy* is exact, that is, this form is the differential

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y} \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

This condition is not only necessary but also sufficient for * to be an exact differential equation.

$$M(x, y)dx + N(x, y)dy = 0 \quad *$$

By integration we immediately obtain the general solution of * in the form

$f(x, y) = c$ is general solution

$Mdx + Ndy$ Exact diff.

$M(x, y)dx + N(x, y)dy = 0$ Exact diff. Eq.

So $\partial M/\partial y=\partial N/\partial x$ is a necessary condition for the exactness of *. We shall prove that is also sufficient by showing that $\partial M/\partial y=\partial N/\partial x$ enables us to construct a function f that satisfy $\partial f/\partial x= M$ $\partial f/\partial y=N$.

$$\frac{\partial f}{\partial x} = M \qquad f = \int M dx + g(y)$$

$$\frac{\partial f}{\partial y} = N \qquad \frac{\partial}{\partial y}\int M dx + g'(y) = N$$

$$\frac{dg}{dy} = N - \frac{\partial}{\partial y}\int M dx \qquad g(y) = \int \left[N - \frac{\partial}{\partial y}\int M dx) \right] dy$$

$$f = \int M dx + \int \left[N - \frac{\partial}{\partial y}\int M dx \right] dy$$

$$\frac{\partial f}{\partial x} = M(x, y) + 0$$

$$\frac{\partial}{\partial x}\left[\int N - \frac{\partial}{\partial y}\int M dx\right] dy \overset{?}{=} 0$$

$$\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy = 0$$

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Example $e^y dx + (xe^y + 2y)dy = 0$

$$M = e^y \quad , \quad N = xe^y + 2y \qquad \frac{\partial M}{\partial y} = e^y = \frac{\partial N}{\partial x}$$

This tells us that there exists a function *f(x,y)* such that

$$\frac{\partial f}{\partial x} = M = e^y \quad f = \int e^y dx + g(y) = xe^y + g(y)$$

$$\frac{\partial f}{\partial y} = xe^y + g'(y) = N = xe^y + 2y \qquad g'(y) = 2y$$

$$g(y) = y^2 + c' \qquad f = xe^y + y^2 = c \quad \text{g.s.}$$

Example

Solve $\cos(x + y)\,dx + (3y^2 + 2y + \cos(x + y))\,dy = 0.$

$M = \cos(x + y)$

$N = 3y^2 + 2y + \cos(x + y)$

$$\frac{\partial M}{\partial y} = -\sin(x + y)$$

$$\frac{\partial N}{\partial x} = -\sin(x + y)$$

$$f = \int M dx = \int \cos(x + y) dx = \sin(x + y) + g(y)$$

$$\frac{\partial f}{\partial y} = \cos(x + y) + \frac{dg}{dy} = N = 3y^2 + 2y + \cos(x + y)$$

Hence

$$dg/dy = 3y^2 + 2y.$$

$$g(y) = \int (3y^2 + 2y)dy = y^3 + y^2 + c$$

$$f(x, y) = \sin(x + y) + y^3 + y^2 = c$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$= \cos(x + y)dx + (\cos(x + y) + 3y^2 + 2y)dy = 0$$

Example

$$(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0$$

$$M = x^3 + 3xy^2$$

$$N = 3x^2y + y^3$$

$$\frac{\partial M}{\partial y} = 6xy = \frac{\partial N}{\partial x}$$

$$(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0 \quad \text{is exact}$$

$$f=\int M dx+g(y)=\int (x^3+3xy^2)dx+g(y)=\frac{1}{4}x^4+\frac{3}{2}x^2y^2+g(y)$$

$$\frac{\partial f}{\partial y}=3x^2y+\frac{dg}{dy}=N=3x^2y+y^3$$

$$\Rightarrow \frac{dg}{dy}=y^3 \Rightarrow g=\frac{y^4}{4}+c'$$

$$f(x,y)=\frac{1}{4}(x^4+6x^2y^2+y^4)=c$$

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

$$(x^3+3xy^2)dx+(3x^2y+y^3)dy=0$$

Non-Exact Equation

$$ydx + (x^2 y - x)dy = 0$$

$$\frac{\partial M}{\partial y} = 1 \;,\quad \frac{\partial N}{\partial x} = 2xy - 1 \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Integrating Factors

It is sometimes possible to convert a differential equation that is not exact into an exact equation by multiplying the equation by a suitable integrating factor $\mu(x, y)$:

$$\frac{y}{x^2} dx + (y - \frac{1}{x})dy = 0$$

$$M(x,y)dx + N(x,y)dy = 0 \quad *$$

$$\mu(M(x,y)dx + N(x,y)dy) = 0$$

$$f(x,y) = c \quad \text{g. s.}*$$

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \quad **$$

$$*,** \rightarrow \frac{dy}{dx} = -\frac{M}{N} = -\frac{\partial f / \partial x}{\partial f / \partial y}$$

$$\frac{\partial f/\partial x}{M} = \frac{\partial f/\partial y}{N} = \mu \qquad \frac{\partial f}{\partial x} = \mu M \;,\; \frac{\partial f}{\partial y} = \mu N$$

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

$$\mu\frac{\partial M}{\partial y} + M\frac{\partial \mu}{\partial y} = \mu\frac{\partial N}{\partial x} + N\frac{\partial \mu}{\partial x}$$

$$\frac{1}{\mu}\left(N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y}\right) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$$

Much more difficult problem respect to *

$$Case\,1)\quad \mu(x),\quad \frac{\partial \mu}{\partial y} = 0 \qquad\qquad \frac{\partial \mu}{\partial x} = \frac{d\mu}{dx}$$

$$\frac{1}{\mu}\left(N\frac{\partial \mu}{\partial x} - M\cancel{\frac{\partial \mu}{\partial y}}\right) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$$

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{\partial M/\partial y - \partial N/\partial x}{N} = g(x)$$

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = g(x)$$

$$\frac{1}{\mu}\frac{d\mu}{dx} = g(x) \qquad \int g(x)dx = \ln\,\mu \qquad \mu = e^{\int g(x)dx}$$

Example $$ydx + (x^2y - x)dy = 0$$

$$\frac{\partial M}{\partial y} = 1 \ , \quad \frac{\partial N}{\partial x} = 2xy - 1$$

$$\frac{\partial M / \partial y - \partial N / \partial x}{N} = \frac{1 - (2xy - 1)}{x^2y - x} = -\frac{2}{x}$$

$$\mu = e^{\int -(2/x)dx} = e^{-2\ln x} = x^{-2}$$

$$ydx + (x^2 y - x)dy = 0$$

$$\frac{y}{x^2}dx + \frac{1}{x^2}(x^2 y - x)dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{x^2}\right) = \frac{1}{x^2} \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{x^2}(x^2 y - x)\right] = \frac{1}{x^2}$$

$$x^2 ydy - (xdy - ydx) = 0 \quad ydy = \frac{xdy - ydx}{x^2}$$

$$ydy - d(y/x) = 0 \quad \frac{y^2}{2} - \frac{y}{x} = c$$

There is another useful technique for converting simple non exact equation into exact ones.

$$ydx + (x^2 y - x)dy = 0 \quad *$$

$$x^2 ydy - (xdy - ydx) = 0 \qquad d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$$

$$ydy - d\left(\frac{y}{x}\right) = 0 \qquad \frac{1}{2}y^2 - \frac{y}{x} = c$$

In effect, we have found an integrating factor for * . The following are some other differential formulas that often useful in similar circumstances:

$$(xdy - ydx) \quad \times \frac{1}{y^2} \qquad d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$$

$$(xdy - ydx) \quad \times \frac{1}{xy} \qquad d\left(\ln \frac{x}{y}\right) = \frac{ydx - xdy}{xy}$$

$$d\left(\tan^{-1} \frac{x}{y}\right) = \frac{ydx - xdy}{x^2 + y^2}$$

$$d(xy) = xdy + ydx$$

$$d(x^2 + y^2) = 2(xdx + ydy)$$

$$\text{Case 2)} \quad \mu(y), \quad \frac{\partial \mu}{\partial x} = 0$$

$$\mu \frac{\partial M}{\partial y} + M \frac{\partial \mu}{\partial y} = \mu \frac{\partial N}{\partial x} + N \cancel{\frac{\partial \mu}{\partial x}}$$

$$\mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -M\left(\frac{d\mu}{dy}\right)$$

$$\frac{1}{\mu}\frac{d\mu}{dy} = \frac{\partial M / \partial y - \partial N / \partial x}{-M} = h(y) \qquad \frac{1}{\mu}\frac{d\mu}{dy} = h(y)$$

$$\int h(y)dy = \ln \mu \qquad \mu = e^{\int h(y)dy}$$

Example $$ydx + (2x - 3y)dy = 0$$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 2$$

$$\frac{\partial M / \partial y - \partial N / \partial x}{-M} = \frac{1}{y} = h(y)$$

$$\mu = e^{\int h(y)dy} = e^{\int \frac{dy}{y}} = e^{\ln y} = y$$

$$y^2 dx + y(2x - 3y)dy = 0$$ **Exact Eq.**

$$\int_{x_0}^{x} M(x, y)dx + \int_{y_0}^{y} N(x_0, y)dy = c$$

$$\int_{x_0}^{x} y^2 dx + \int_{y_0}^{y} 2x_0 y dy - \int_{y_0}^{y} 3y^2 dy = c'$$

$$xy^2 - x_0 y^2 + x_0 y^2 - x_0 y_0{}^2 - y^3 + y_0{}^3 = c'$$

$$xy^2 - y^3 = c$$

Thanks for your attention