Differential Equations

Lecture 5

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Chapter 2 – First Order Equations

$$F(x, y, y') = 0 \qquad \text{General Form}$$

$$\frac{dy}{dx} = y' = f(x, y)$$

In general, solving a differential equation is not an easy matter.

Separable equation

A *separable equation* is a first-order differential equation that can be written in the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

The name *separable* comes from the fact that the expression on the right side can be "separated" into a function of x and a function of y.

To solve this equation we rewrite it in the differential form

$$h(y)dy = g(x)dx$$

$$\int h(y)dy = \int g(x)dx$$

It defines y implicitly as a function of x. In some cases we may be able to solve for y in terms of x

Example 1: Solve the differential equation $\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}$

$$(2y + \cos y)dy = 6x^2 dx$$

$$\int (2y + \cos y)dy = \int 6x^2 dx$$

$$y^2 + \sin y = 2x^3 + c$$

$$c = c_2 - c_1$$

The above general solution is in implicit form. In this case it is impossible to express *y* explicitly as a function of *x*.

Example 2: Solve the differential equation $y' = x^2 y$

$$\frac{dy}{dx} = x^2 y \qquad \frac{dy}{y} = x^2 dx$$

If $y \neq 0$, we can rewrite it in differential notation and integrate:

$$\ln y = \frac{x^3}{3} + c$$

$$y = e^{\frac{x^3}{3} + c} = e^c e^{x^3/3}, \quad y = Ce^{x^3/3}$$

Example 3 : Differential equation of the form $y' = g(\frac{y}{x})$

The form of the equation suggests that we set $\frac{y}{x} = z$

$$y = zx \qquad y' = z'x + z$$

$$y' = g(\frac{y}{x})$$

$$z'x = g(z) - z \qquad \frac{dz}{g(z) - z} = \frac{dx}{x}$$

First-Order Homogeneous Equations

A function $f(x,y)$ is said to be homogeneous of degree n if the equation

$$f(tx,ty)=t^n f(x,y)$$

Example 1:

The function $f(x,y) = x^2 + y^2$ is homogeneous of degree 2, since

$$f(tx,ty)=(tx)^2+(ty)^2=t^2(x^2+y^2)=t^2 f(x,y)$$

Example 2:

The function $f(x,y) = \sin(x/y)$ is homogeneous of degree 0, since

$$f(tx,ty)=\sin(tx/ty)=t^0\sin(x/y)=t^0 f(x,y)$$

Example 3:

The function $f(x,y)=\sqrt{x^8-3x^2y^6}$ **is homogeneous of degree 4, since**

$$f(tx,ty)=\sqrt{(tx)^8-3(tx)^2(ty)^6}=\sqrt{t^8(x^8-3x^2y^6)}$$

$$=\sqrt{t^8}\sqrt{x^8-3x^2y^6}=t^4f(x,y)$$

Example 4:

The function ***f***(*x*,*y*) = 2 *x* + *y* **is homogeneous of degree 1, since**

$$f(tx,ty)=2(tx)+(ty)=t(2x+y)=t^1f(x,y)$$

Example 5:

The function $f(x,y) = x^3 - y^2$ is not homogeneous, since

$$f(tx, ty) = (tx)^3 - (ty)^2 = t^3x^3 - t^2y^2$$

which does not equal $t^n f(x,y)$ for any n.

first-order differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is said to be homogeneous if *M*(*x,y*) and *N*(*x,y*) are both homogeneous functions of the same degree.

This equation can then be written in the form $dy/dx = f(x,y)$

$$where \quad f(x,y) = -\frac{M(x,y)}{N(x,y)} \quad is \ \text{hom}ogeneous\ of\ \text{deg}ree\ 0$$

$$\text{solving the equation} \qquad z = y/x$$

$$f(tx,ty) = t^0 f(x,y) = f(x,y)$$

$$t = 1/x \qquad f(x,y) = f(1, y/x) = f(1,z)$$

$$y = zx \rightarrow \frac{dy}{dx} = z + x\frac{dz}{dx}$$

$$dy/dx = f(x, y)$$

$$z + x\frac{dz}{dx} = f(1, z)$$

$$\frac{dz}{f(1, z) - z} = \frac{dx}{x}$$

We now complete the solution by integrating and replacing z by y/x.

Example: $(x+y)dx-(x-y)dy=0$

$$M(x,y)=(x+y)\rightarrow(tx,ty)=t(x,y)=tM(x,y)$$

$$N(x,y)=-(x-y)\rightarrow-(tx,ty)=-t(x,y)=-tN(x,y)$$

$$\frac{dy}{dx}=\frac{(tx+ty)}{(tx-ty)} \qquad \frac{dy}{dx}=\frac{1+y/x}{1+y/x}=\frac{1+z}{1-z}$$

$$z=\frac{y}{x}\rightarrow y=zx\rightarrow\frac{dy}{dx}=x\frac{dz}{dx}+z$$

$$x\frac{dz}{dx}+z=\frac{1+z}{1-z}$$

$$\frac{(1-z)dz}{1+z^2} = \frac{dx}{x} \qquad \int \frac{dz}{1+z^2} - \int \frac{zdz}{1+z^2} = \int \ln x$$

$$\tan^{-1} z - \frac{1}{2}\ln(1+z^2) = \ln x + c$$

$$\tan^{-1}\frac{y}{x} = \ln x\sqrt{1+(y/x)^2} + c$$

$$\tan^{-1}\frac{y}{x} = \ln\sqrt{x^2+y^2} + c$$

Example : The differential equation

$$(x^2 - y^2)dx + xydy = 0$$

is homogeneous because both *M*(*x,y*) = $x^2 - y^2$ and *N*(*x,y*) = *xy* are homogeneous functions of the same degree (namely, 2).

Example : Solve the equation

This equation is homogeneous, as observed in above Example . Thus to solve it, make the substitutions

$$y = xz \rightarrow dy = xdz + zdx$$

$$\left[x^2 - (xz)^2\right]dx + \left[x(xz)\right](xdz + zdx) = 0$$

$$(x^2 - x^2z^2)dx + x^3zdz + x^2z^2dx = 0$$

$$x^2dx + x^3zdz = 0 \rightarrow dx + xzdz = 0$$

This final equation is now separable $zdz = -\frac{dx}{x}$

$$\int zdz = \int -\frac{dx}{x}$$

$$\frac{1}{2}z^2 = -\ln x + c' \qquad \frac{1}{2}z^2 = \ln\frac{c}{x}$$

$$\frac{1}{2}\left(\frac{y}{x}\right)^2 = \ln\frac{c}{x} \Rightarrow y^2 = 2x^2\ln\frac{c}{x}$$

Expansion of Total Derivative

If $f = f\left(x, y, z; t\right)$ then

$$\delta f = \left(\frac{\partial f}{\partial t}\right)\delta t + \left(\frac{\partial f}{\partial x}\right)\delta x + \left(\frac{\partial f}{\partial y}\right)\delta y + \left(\frac{\partial f}{\partial z}\right)\delta z + H.O.T$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$u \equiv \frac{dx}{dt}, \quad v \equiv \frac{dy}{dt}, \quad w \equiv \frac{dz}{dt}$$

u = west-east component of fluid velocity

v = south-north component of fluid velocity

w = vertical component of fluid velocity

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}$$

A B C D E

Term A: Total rate of change of *f* following the fluid motion

Term B: Local rate of change of *f* at a fixed location

Term C: Advection of *f* in *x* direction by the *x*-component flow

Term D: Advection of *f* in *y* direction by the *y*-component flow

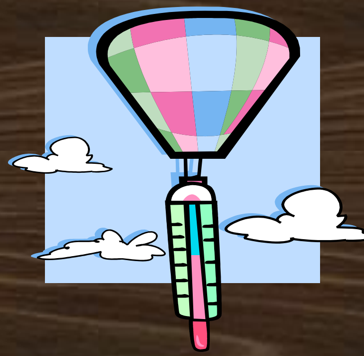
Term E: Advection of *f* in *z* direction by the *z*-component flow

Total Derivative vs. Local Derivative

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{U}.\nabla f \qquad \frac{\partial f}{\partial t} = \frac{df}{dt} - \vec{U}.\nabla f$$

Total derivative is the temporal rate of change following the fluid motion.

Example: A thermometer measuring changes as a balloon floats through the atmosphere.



Local derivative is the temporal rate of Change at a fixed point.

Example: An observer measures changes in temperature at a weather station.

Exact Differential Equations

If we have a family of curves *f(x,y)=c*, then its differential equation can be written in the form:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

For example, the family $x^2y^3=c$ has:

$$2xy^3dx + 3x^2y^2dy = 0$$

Suppose we turn this situation around, and begin with the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

A first –order differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

If there happens to exist a function *f(x,y)* such that:

$$\frac{\partial f}{\partial x} = M \quad \text{and} \quad \frac{\partial f}{\partial y} = N$$

is called an exact differential equation if the differential form *M*(*x*, *y*) *dx* + *N*(*x*, *y*) *dy* is exact, that is, this form is the differential

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y} \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

This condition is not only necessary but also sufficient for * to be an exact differential equation.

$$M(x, y)dx + N(x, y)dy = 0 \quad *$$

By integration we immediately obtain the general solution of * in the form

$f(x, y) = c$ is general solution

$Mdx + Ndy$ Exact diff.

$M(x, y)dx + N(x, y)dy = 0$ Exact diff. Eq.

Example

$$e^{y}dx + (xe^{y} + 2y)dy = 0$$

$$M = e^{y} \quad , \quad N = xe^{y} + 2y$$

$$\frac{\partial M}{\partial y} = e^{y} \quad , \quad \frac{\partial N}{\partial x} = e^{y}$$

Thanks for your attention