معادلات دیفرانسیل

درس چهارم

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Explicit function:

If y is completely defined in terms of x, y is called an explicit. $y = x^2 - 4x + 2$

Implicit function

$x^2 + y^2 = 25$ $\quad 2x + 2y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}.$

$x^2 + 2xy + 3y^2 = 4$

$$2x + (2x\frac{dy}{dx} + 2y) + 6y\frac{dy}{dx} = 0 \Rightarrow (2x+6y)\frac{dy}{dx} = -2x - 2y$$

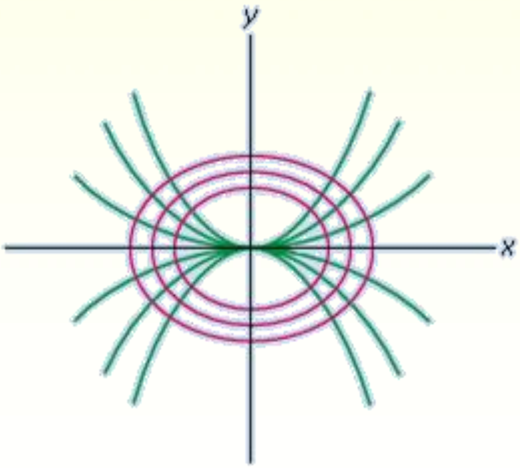
Standard Derivative

	$y=f(x)$	$\frac{df}{dx}$
1	x^n	nx^{n-1}
2	e^x	e^x
3	e^{kx}	ke^{kx}
4	a^x	$a^x \cdot \ln a$
5	$\ln x$	$\frac{1}{x}$
6	$\log_a x$	$\frac{1}{x \cdot \ln a}$
7	$\sin x$	$\cos x$
8	$\cos x$	$-\sin x$
9	$\tan x$	$\sec^2 x$
10	$\cot x$	$-\mathrm{cosec}^2 x$
11	$\sec x$	$\sec x \cdot \tan x$
12	$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x \cdot \cot x$
13	$\arcsin x$ ($\sin^{-1} x$)	$\frac{1}{\sqrt{1-x^2}}$
14	$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
15	$\arctan x$	$\frac{1}{1+x^2}$
16	$\mathrm{arccot}\, x$	$-\frac{1}{1+x^2}$
17	$\sinh x=(e^x+e^{-x})/2$	$\cosh x$
18	$\cosh x==(e^x-e^{-x})/2$	$\sinh x$

$$y = cx^2 \qquad dy/dx = 2cx \qquad \frac{dy}{dx} = \frac{2xy}{x^2} = \frac{2y}{x}$$

$$-\left(\frac{dx}{dy}\right) = \frac{2y}{x} \qquad 2\int y\,dy = -\int x\,dx \qquad y^2 = -\frac{x^2}{2} + c$$

$$\frac{y^2}{c} + \frac{x^2}{2c} = 1$$



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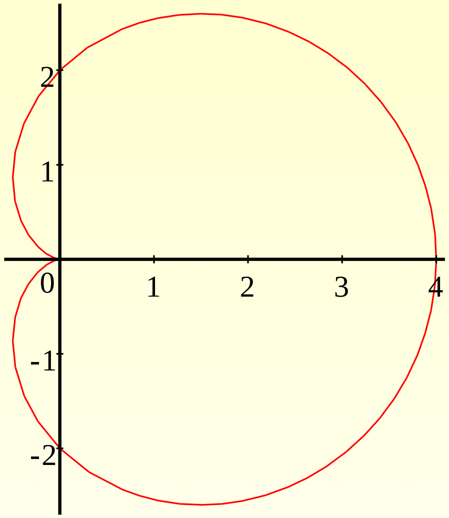

$$r = 2(1 + \cos\theta)$$

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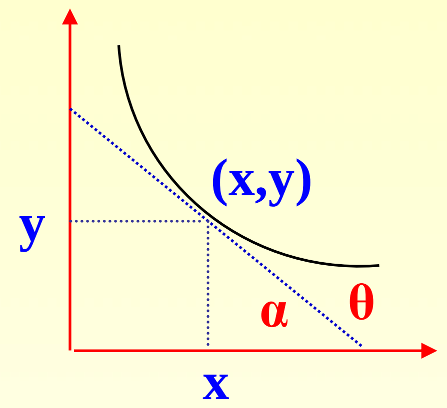
$$m=\left(\frac{dy}{dx}\right)_{(x,y)}=\mathrm{tg}\theta=-\mathrm{tg}\alpha=-\frac{\mathrm{y}}{\mathrm{x}}$$



$$\frac{dy}{dx}=-\frac{y}{x}$$

$$\ln y=-\ln x+\ln c$$

$$xy=c$$ **Hyperbola family**

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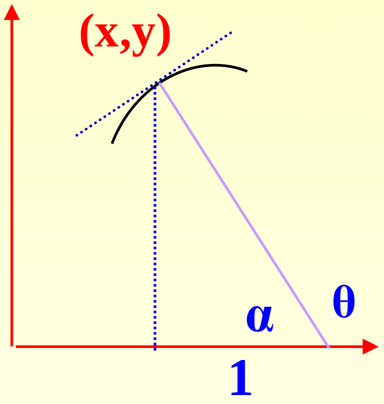


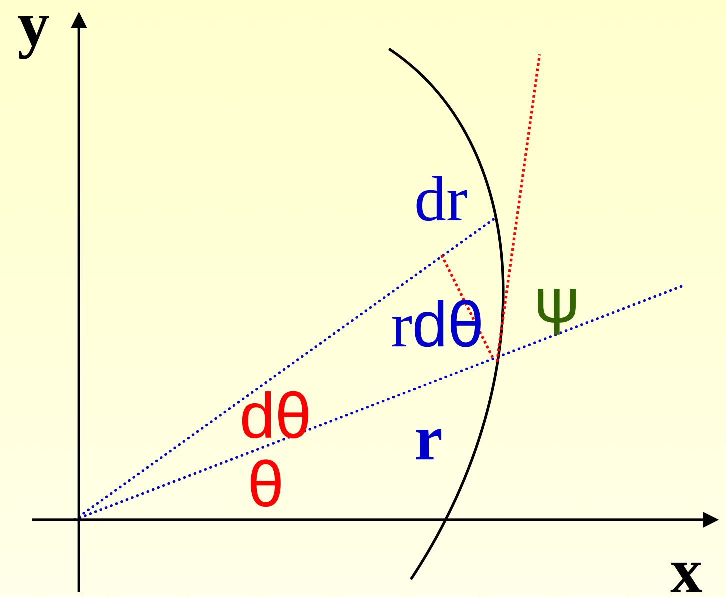
$$tg\alpha = \frac{y}{1} = -tg\theta = -(-\frac{dx}{dy})$$

$$ydy = dx \rightarrow y^2 = 2x + c$$

$$c)\ tg\alpha = \frac{y}{1} = \frac{dy}{dx} \rightarrow y = ce^x$$

d) $\psi = \theta$

The angle between the tangent and radial line at the point (r, θ) is



$$tg\psi = \frac{dy}{dx} = \frac{rd\theta}{dr}$$

$$\psi = \theta \rightarrow tg\theta = \frac{rd\theta}{dr}$$

$$r = c\sin\theta$$

e) $\psi = c \rightarrow \frac{rd\theta}{dr} = c$

Page 20-5: Newton's Law of Cooling

The rate of cooling of an object is directly proportional to the temperature difference between the object and its surroundings.

$$\frac{dT}{dt} = k(T - T_s)$$

The Problem

If an object takes 40 minutes to cool from 30 degrees to 24 degrees in a 20 degree room, how long will it take the object to cool to 21 degrees?

Solving the Differential Equation

$$\frac{dT}{dt} = k(T-20) \Rightarrow \frac{dT}{T-20} = k\,dt \Rightarrow$$

$$\int \frac{dT}{T-20} = \int k\,dt \Rightarrow \ln(T-20) = kt + c \Rightarrow$$

$$T - 20 = e^{(kt+c)} = c'e^{kt} \Rightarrow$$

$$T = 20 + c'e^{kt}$$

$$c' = ?$$

$$T(0) = 20 + c'e^{k\times 0} = 20 + c'$$

So c' is the difference between the initial temperature of the cooling body and the surrounding room.

Since the initial temperature of the body is 30 degrees, we have

$$c' = 30 - 20 = 10$$

$$T = 20 + 10e^{kt}$$

K=?

Since it takes 40 minutes for the body to cool from 30 degrees to 24 degrees we have

$$T(40) = 24 = 20 + 10e^{40k} \Rightarrow 4 = 10e^{40k} \Rightarrow$$

$$\ln(4) = \ln(10e^{40k}) = \ln(10) + \ln(e^{40k})$$

$$\ln(4) - \ln(10) = \ln(e^{40k})$$

$$\ln(2/5) = 40k \Rightarrow k = \frac{\ln(2/5)}{40} = -0.023$$

$$T = 20 + 10e^{kt} \qquad T = 20 + 10e^{-0.023t}$$

How Long to Cool to 21?

We want *t* when *T*=21 so

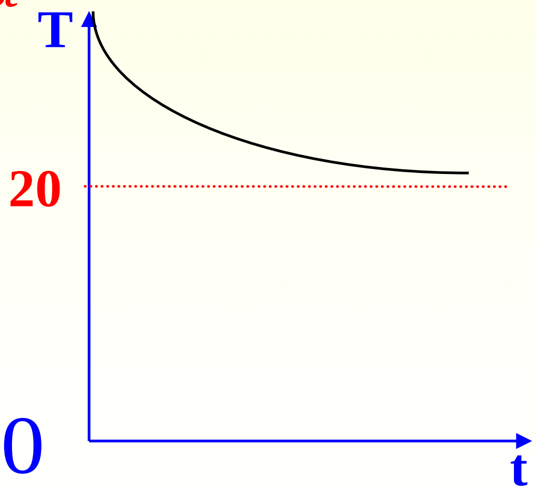
$$T = 21 = 20 + 10e^{-0.023t}$$

$$1 = 10e^{-0.023t} \Rightarrow 1/10 = e^{-0.023t}$$

$$\ln(1/10) = -0.023t$$

$$t = 100.1\,\text{min}$$

$$\lim_{t\to\infty} T(t) = \lim_{t\to\infty}(20 + 10e^{-0.023t}) = 20$$



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$$\frac{dv}{dt} = g - cv^2 \qquad c = k/m$$

$$v = f(t) \qquad t = 0, \quad v = 0$$

$$m\frac{dv}{dt} = mg - kv^2 \rightarrow \frac{dv}{dt} = -\frac{k}{m}(v^2 - \frac{gm}{k})$$

$$b^2 = \frac{mg}{k} \qquad \frac{dv}{dt} = -c(v^2 - b^2)$$

$$\frac{dv}{v^2 - b^2} = -cdt$$

$$\frac{1}{v^2-b^2}=\frac{1}{2b}\left(\frac{1}{v-b}-\frac{1}{v+b}\right)$$

$$\int\frac{dv}{v^2-b^2}=\frac{1}{2b}\int\frac{dv}{v-b}-\frac{1}{2b}\int\frac{dv}{v+b}=-\int\frac{k}{m}dt$$

$$=\frac{1}{2b}\ln(v-b)-\frac{1}{2b}\ln(v+b)=-\frac{k}{m}t+c'$$

$$=\frac{1}{2b}\ln\frac{v-b}{v+b}=-\frac{k}{m}t+c'$$

$$\frac{v-b}{v+b}=ce^{-\frac{2kb}{m}t} \qquad c=e^{2c'b}$$

$$v-b=(v+b)ce^{-\frac{2kb}{m}t}$$

$$v(1-ce^{-\frac{2kb}{m}t})=b(1+ce^{-\frac{2kb}{m}t})$$

$$v(t)=b\frac{1+ce^{-pt}}{1-ce^{-pt}} \qquad v(0)=0=b\frac{1+c}{1-c}$$

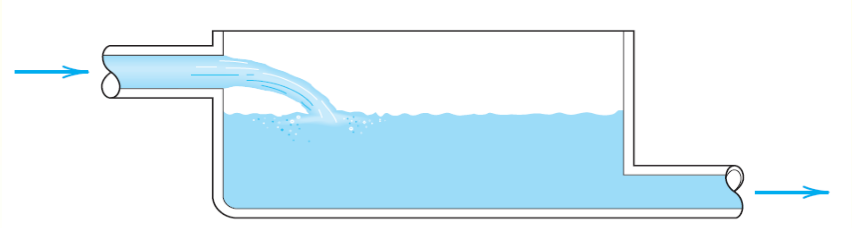
$$c = -1 \qquad v(t) = b\frac{1 + ce^{-pt}}{1 - ce^{-pt}}$$

$$v(t) = b\frac{1 - e^{-pt}}{1 + e^{-pt}} \qquad v(t) = \sqrt{\frac{mg}{k}}\,\frac{1 - e^{-2\sqrt{\frac{kg}{m}}t}}{1 + e^{-2\sqrt{\frac{kg}{m}}t}}$$

$$v_t = \lim_{t \to \infty} v(t) = \sqrt{\frac{mg}{k}}$$

Page 35-3: Mixing problem

A tank contains 20kg of salt dissolved in 5000L of water. Brine that contains 0.03kg of salt per liter of water enters the tank at the rate of 25L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?



$y(t)$: the amount of salt (in kilograms) after t minutes.

$y(0) = 20$ and we want to find $y(30)$.

Note that dy/dt is the rate of change in the amount of salt

dy/dt = Salt inflow rate – Salt outflow rate

rate in = (0.03kg/L)(25L/min) = 0.75 kg/min

The tank always contains 5000L of liquid, so the concentration at time t is $y(t)/5000$ (kg/L).

Since the brine flows out at a rate of 25L/min

rate out = ($y(t)/5000$ kg/L)(25L/min) = [$y(t)/200$]kg/min

dy/dt = Salt inflow rate – Salt outflow rate

$$\frac{dy}{dt} = 0.75 - \frac{y(t)}{200} = \frac{150 - y(t)}{200}$$

$$\int \frac{dy}{150 - y} = \int \frac{dt}{200} \qquad -\ln|150 - y| = \frac{t}{200} + c$$

Since $y(0) = 20$ → **$-\ln 130 = c$**

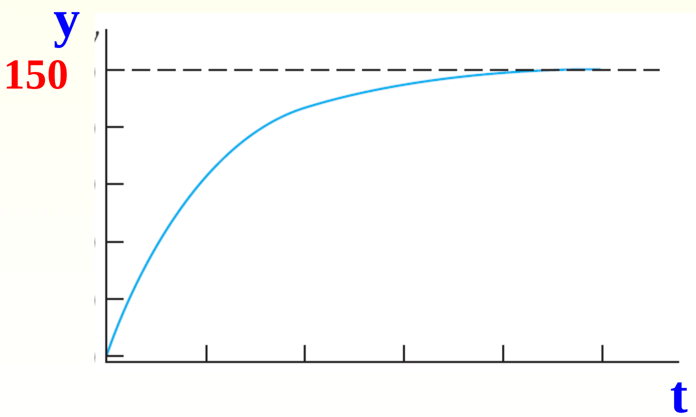
$$-\ln|150 - y| = \frac{t}{200} - \ln 130$$

$$150 - y = 130e^{-t/200}$$

$$y(t) = 150 - 130e^{-\frac{t}{200}}.$$

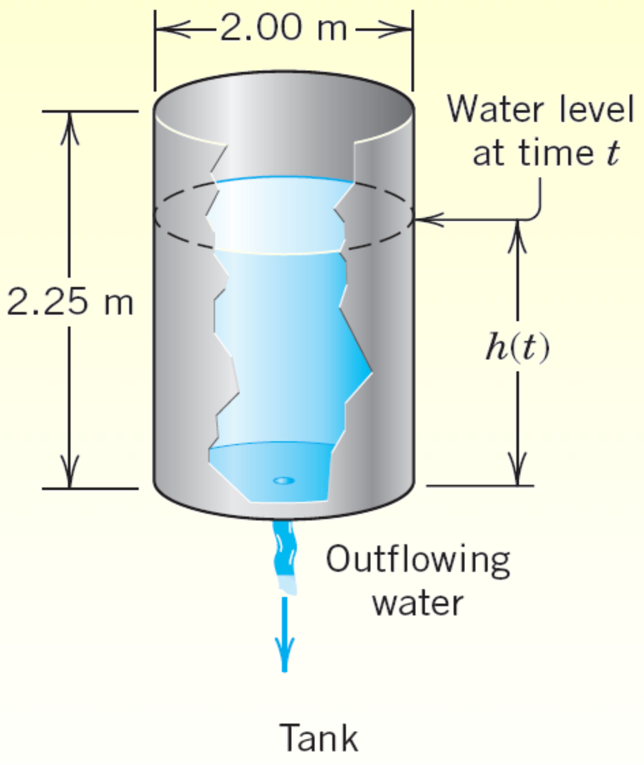
The amount of salt after 30 min is

$$y(30) = 150 - 130e^{-\frac{30}{200}} \approx 38.1\,\text{kg}.$$



Page 35-7: Leaking Tank. Outflow of Water Through a Hole (Torricelli's Law)

This is another prototype engineering problem that leads to an ODE. It concerns the outflow of water from a cylindrical tank with a hole at the bottom. You are asked to find the height of the water in the tank at any time if the tank has diameter 2 m, the hole has diameter 1 cm, and the initial height of the water when the hole is opened is 2.25 m. When will the tank be empty?



Physical information. **Under the influence of gravity the outflowing water has velocity:**

$$v(t) = 0.6\sqrt{2gh(t)}$$ **(Torricelli's law),**

where $h(t)$ is the height of the water above the hole at time t, and g *is* acceleration of gravity at the surface of the earth.

First we look at the amount of water that is running <u>out of the tank</u> in a time interval dt

To get a D. E. for $h(t)$, we have to consider that the volume V of the water running out in an interval dt is

$$dV_{out} = A.v(t).dt$$ **(A is the area of the hole.)**

This volume must be equal to that the water missing in the tank

$$-dV_{in} = B.dh$$ **(B is the cross-section of the tank)**

$$-B.dh = A.v(t).dt$$

$$\frac{dh}{dt} = -\frac{A}{B}v(t) = -\frac{A}{B}0.6\sqrt{2gh(t)}$$

$$\frac{dh}{dt} = -26.56\frac{A}{B}\sqrt{h}$$

Means: the change in the water level is proportional to it's square root!

$$\frac{dh}{\sqrt{h}} = -26.56\frac{A}{B}dt \qquad 2\sqrt{h} = c' - 26.56\frac{A}{B}t$$

$$h = (c - 13.28At/B)^2$$

$$h_g(t) = (c - 0.000332t)^2$$

Use initial values to find particular solution:

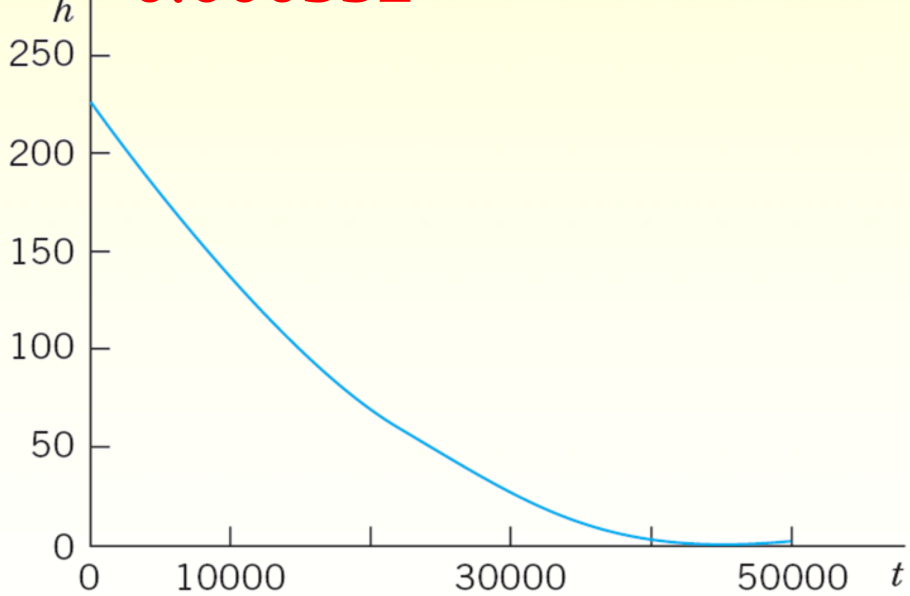
The initial height (the initial condition) is *h*(0) = 225 cm

$$c^2 = 225 \rightarrow c = 15 \qquad h_p(t) = (15 - 000332t)^2$$

$$h_p(t) = (15 - 000332t)^2$$

Tank empty

$$h_p(t) = 0 \rightarrow t = \frac{15}{0.000332} = 45181s = 12.6h$$



Water level $h(t)$ in tank

Thanks for your attention

$$r = 2\cos(2\theta)$$