معادلات دیفرانسیل

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Review of Lesson 2

Nonhomogeneous Differential Equation

General Remarks on Solutions

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0 \rightarrow y = y(x, c_1, c_2, \ldots, c_n)$$

General Solution

Particular Solution

The geometric meaning of a solution

Families of Curves. Orthogonal Trajectories

Linear Equation Applications

Falling Bodies and Other Rate problems

$$F = ma \quad , \quad mg = m\frac{d^2y}{dt^2}$$

$$\frac{d^2y}{dt^2} = g \quad , \quad \frac{\mathrm{dv}}{\mathrm{dt}} = g$$

$$dv = gdt \quad , \quad \mathrm{v} = \mathrm{gt} + \mathrm{c}_1$$

$$t = 0 \; , \; v = v_0 \;\rightarrow\; c_1 = v_0 \quad v = gt + v_0$$

$$\frac{dy}{dt} = gt + v_0 \quad y_g = \frac{1}{2}gt^2 + v_0 t + c_2$$

$$t = 0 \;,\; \mathrm{y} = \mathrm{y}_0 \;,\; c_2 = y_0$$

$$y_g = \frac{1}{2} g t^2 + v_0 t + y_0 \qquad v = gt + v_0$$

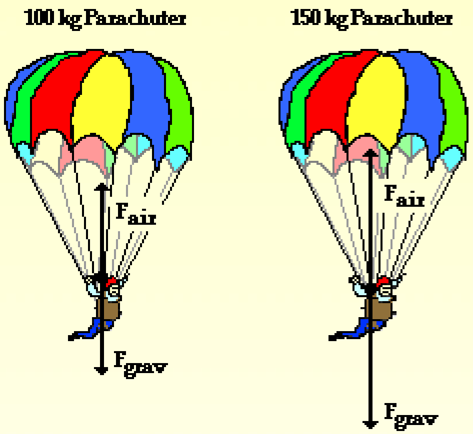
If the body falls from rest starting at $\mathrm{y} = 0$

so that $v_0 = y_0 = 0$

$$y_p = \frac{1}{2} g t^2 \qquad \text{and} \qquad v_p = gt$$

$$v = \sqrt{2gy}$$

Retarded fall



$$F = ma \quad , \quad \mathrm{m}\frac{\mathrm{d}^2 y}{dt^2} = mg - kv$$

$$\frac{d^2 y}{dt^2} = g - cv \quad , \; k/m = c$$

$$\frac{dv}{dt} = g - cv \qquad \frac{dv}{g - cv} = dt$$

$$-\frac{1}{c}\ln(g - cv) = t + c_1 \qquad \ln(g - cv) = -ct - cc_1$$

$$\ln(g-cv)=-ct+c_2 \qquad c_2=-cc_1$$

$$t=0\leftrightarrow v=0\rightarrow c_2=\ln g$$

$$\ln\frac{g-cv}{g}=-ct \qquad g-cv=ge^{-ct}$$

$$v=\frac{g}{c}(1-e^{-ct})=\frac{mg}{k}(1-e^{-\frac{k}{m}t})$$

$$if\ \ t\rightarrow\infty\ ,\ e^{-ct}=0\ \Rightarrow v_t\rightarrow\frac{mg}{k}$$

$$\frac{dy}{dt}=\frac{mg}{k}(1-e^{-\frac{k}{m}t}) \quad dy=\frac{mg}{k}dt-\frac{mg}{k}e^{-\frac{k}{m}t}dt$$

$$y=\frac{mg}{k}t+\frac{m^2g}{k^2}e^{-\frac{k}{m}t}+c_3$$

$$t=0\leftrightarrow y=0\rightarrow c_3=-\frac{m^2g}{k^2}$$

$$y=\frac{mg}{k}t+\frac{m^2g}{k^2}e^{-\frac{k}{m}t}-\frac{m^2g}{k^2}$$

$$y=\frac{mg}{k}t+\frac{m^2g}{k^2}(e^{-\frac{k}{m}t}-1)$$

y as a function of t

The motion of a pendulum

$$mgh_1 = mgh_2 + \frac{1}{2}mv^2$$

$$2gh_1 = 2gh_2 + v^2$$

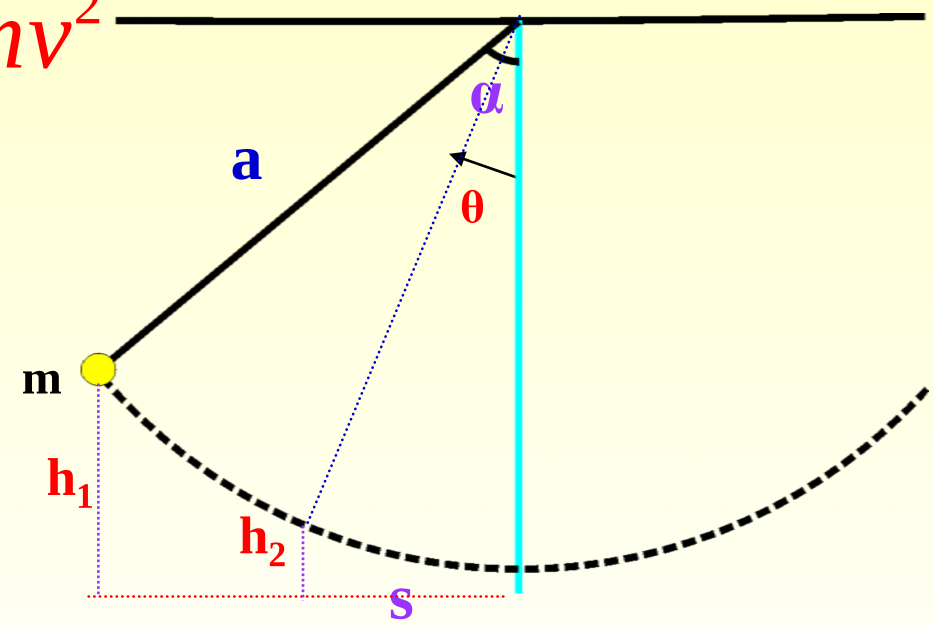
$$2g(h_1 - h_2) = v^2$$

$$h_1 = a - a\cos\alpha$$

$$h_2 = a - a\cos\theta$$

$$2ga(\cos\theta - \cos\alpha) = v^2$$

since $s = a\theta \rightarrow v = ds/dt = a(d\theta/dt)$



$$2ga(\cos\theta - \cos\alpha) = a^2(d\theta / dt)^2$$

$$dt = -\sqrt{\frac{a}{2g}}\frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}$$

$$\int_0^{T/4} dt = \int_0^{\alpha}\sqrt{\frac{a}{2g}}\frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}$$

$$\frac{T}{4} = \sqrt{\frac{a}{2g}}\int_0^{\alpha}\frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}$$

$$\begin{cases} \cos\theta = 1 - 2\sin^2\dfrac{\theta}{2} \\ \cos\alpha = 1 - 2\sin^2\dfrac{\alpha}{2} \end{cases}$$

$$T = 2\sqrt{\frac{a}{g}}\int_0^{\alpha}\frac{d\theta}{\sqrt{\sin^2(\alpha/2) - \sin^2(\theta/2)}}$$

$$\sin(\alpha/2) = k \quad , \quad \sin(\theta/2) = k\sin\varphi$$

$$\cos\frac{\theta}{2} = \sqrt{1-\sin^2\frac{\theta}{2}} \qquad \cos\frac{\theta}{2} = \sqrt{1-k^2\sin^2\varphi}$$

$$\frac{1}{2}\cos\frac{\theta}{2}d\theta = k\cos\varphi d\varphi \to d\theta$$

$$d\theta = \frac{2k\cos\varphi d\varphi}{\cos(\theta/2)} = \frac{2k\cos\varphi d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

$$T = 2\sqrt{\frac{a}{g}}\int_0^{\alpha}\frac{d\theta}{\sqrt{\sin^2(\alpha/2)-\sin^2(\theta/2)}}$$

$$T = 4\sqrt{\frac{a}{g}}\int_0^{\pi/2}\frac{k\cos\varphi d\varphi}{\sqrt{1-k^2\sin^2\varphi}}\times\frac{1}{\sqrt{k^2-k^2\sin^2\varphi}}$$

$$T = 4\sqrt{\frac{a}{g}}\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = 4\sqrt{\frac{a}{g}}F\left(k,\frac{\pi}{2}\right)$$

$$F(k,\varphi)=\int_0^{\varphi}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

The elliptic integral of the first kind

$$E(k,\varphi)=\int_0^{\varphi}\sqrt{1-k^2\sin^2\varphi}\,d\varphi$$

The elliptic integral of the second kind

$$T=4\sqrt{\frac{a}{g}}F\left(k,\frac{\pi}{2}\right)\xrightarrow{k=0}T=4\sqrt{\frac{a}{g}}\times\frac{\pi}{2}$$

$$T=2\pi\sqrt{\frac{a}{g}}$$

Problems pages 25, 26

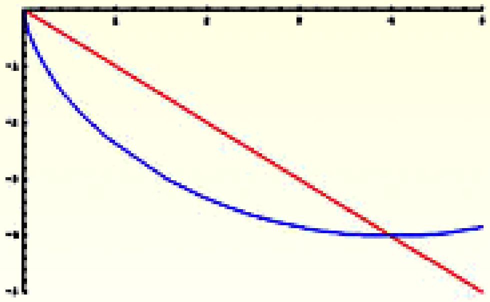
Brachistochrone problem

The Brachistochrone

BRACHIS = SHORT
CHRONOS = TIME

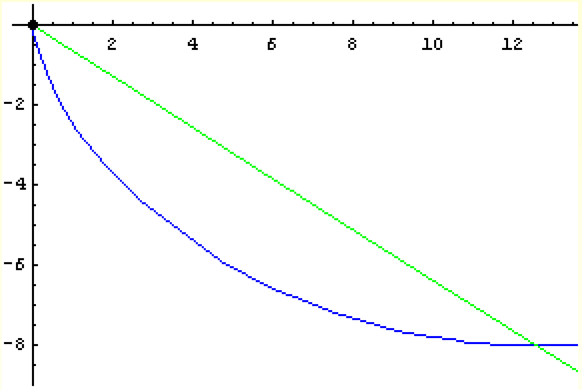
What is the path - curve - producing the shortest possible time for a particle to descend from a given point to another Point.

The shortest distance between two points is a line, but the descent of a weighted particle is acted upon, in the very least, by gravity.



The correct answer is that a body takes less time to fall along the arc of a circumference than to fall along the "line" of a corresponding chord.

The cycloid path allows the particle to move rapidly at first, while in steep descent, and thus build up sufficient speed to overcome the greater distance the particle must travel. Thus, the speed of the descending particle is accelerated by gravity.

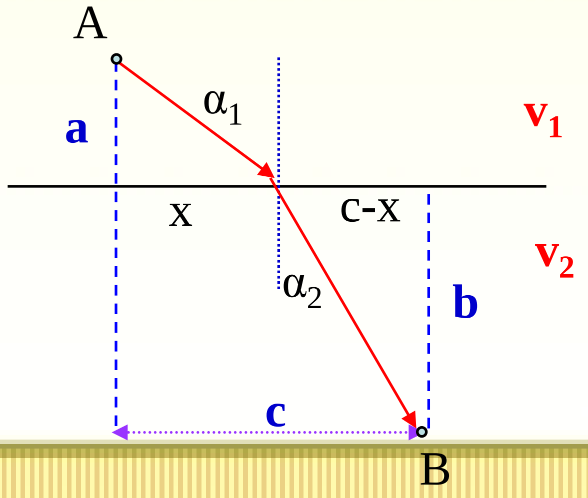


The total time T required for the journey is given by:

$$T = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (c-x)^2}}{v_2}$$

$$dT/dx = 0$$

$$\frac{x}{v_1\sqrt{a^2 + x^2}} = \frac{c-x}{v_2\sqrt{b^2 + (c-x)^2}}$$

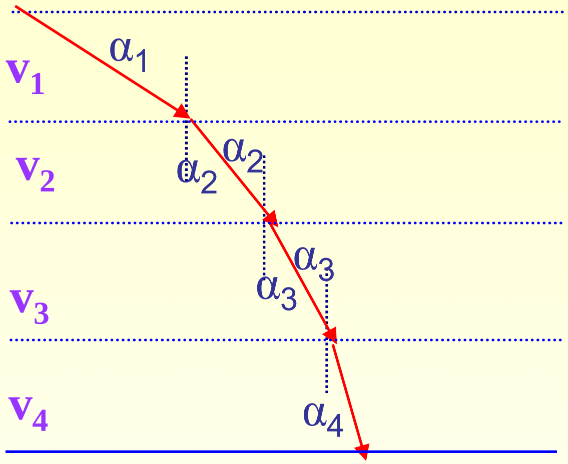


$$\frac{\sin \alpha_1}{v_1} = \frac{\sin \alpha_2}{v_2}$$

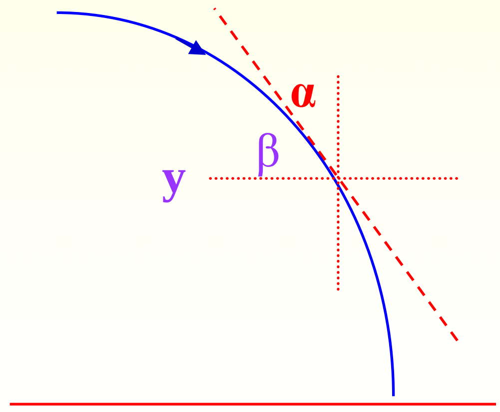
Snell's law of refraction

$$\sin \alpha_1 / \sin \alpha_2 = a \cos \tan t$$

$$\frac{\sin \alpha_1}{v_1} = \frac{\sin \alpha_2}{v_2} = \frac{\sin \alpha_3}{v_3} = \frac{\sin \alpha_4}{v_4}$$



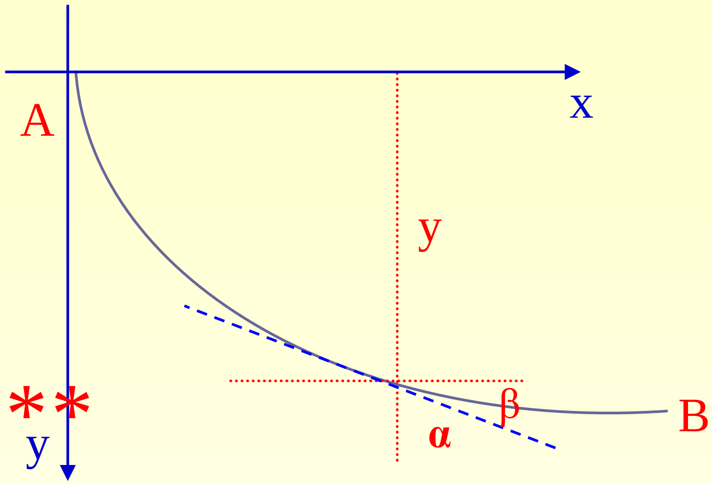
$$\frac{\sin \alpha}{v} = a\ \ cons \tan t$$



Returning now to Bernoulli's problem

$$\frac{\sin\alpha}{v} = c \quad *$$

$$\frac{1}{2}mv^2 = mgy \rightarrow v = \sqrt{2gy} \quad **$$



$$\sin\alpha = \cos\beta = \frac{1}{\sec\beta} = \frac{1}{\sqrt{1+\tan^2\beta}} = \frac{1}{\sqrt{1+(y')^2}} \quad ***$$

$$y\left[1+(y')^2\right] = c$$

The differential equation of the brachistochrone

We now complete our discussion and discover what curve the brachis-tochrone actully is by solving this equation

$$dx = \left(\frac{y}{c-y}\right)^{1/2} dy$$

$$dx = \left(\frac{y}{c-y}\right)^{1/2} dy \qquad \left(\frac{y}{c-y}\right)^{1/2} = \tan\varphi \quad *$$

so that $y = c\sin^2\varphi$ $\quad dy = 2c\sin\varphi\cos\varphi d\varphi$

$$dx = \tan\varphi dy = 2c\sin^2\varphi d\varphi = c(1-\cos 2\varphi)d\varphi$$

Integration now yield

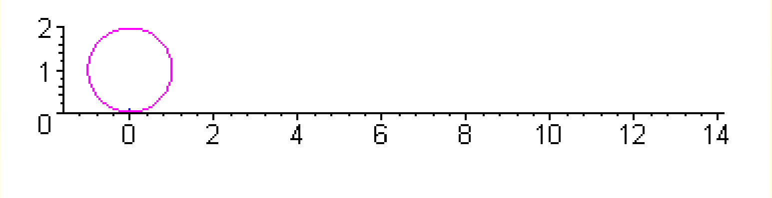
$$x = \frac{c}{2}(2\varphi - \sin 2\varphi) + c_1$$

Our curve is to pass through the origin so by *

$$x = y = 0 \quad \text{when} \quad \varphi = 0 \rightarrow c_1 = 0$$

$$x = \frac{c}{2}(2\varphi - \sin 2\varphi)$$

$$y = c\sin^2\varphi = \frac{c}{2}(1-\cos 2\varphi)$$

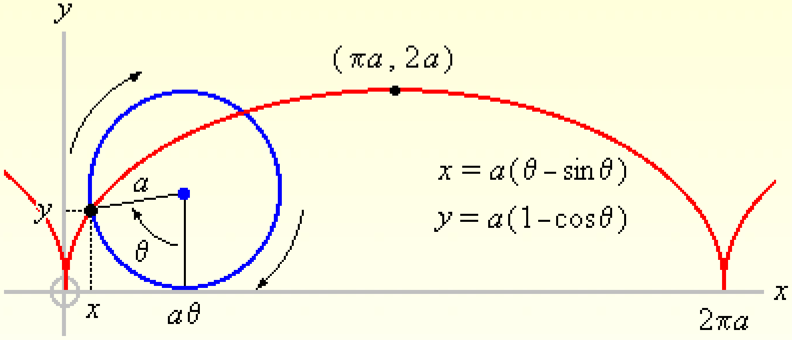


$$a = c/2\ ,\ \theta = 2\varphi$$

So the parametric equations of the cycloid are

$$x = a(\theta - \sin\theta)$$

$$y = a(1-\cos\theta)$$



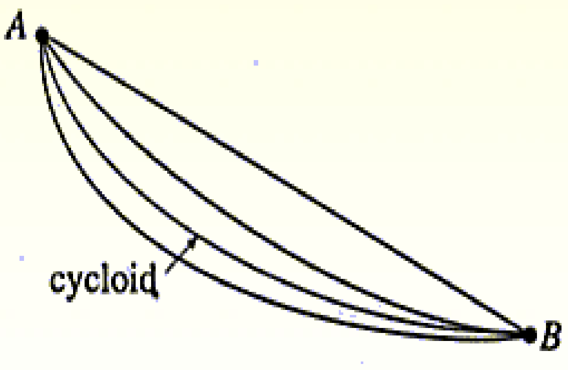
A cycloid is the locus of a point on the circumference of a circle rotating along a fixed line . . . .

the curve of shortest time is a cycloid

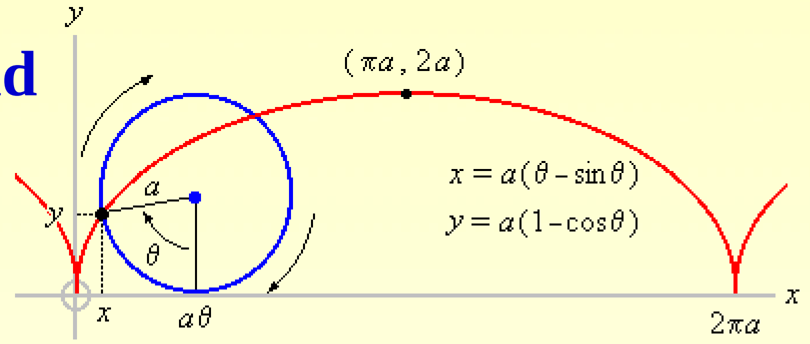
One arch of the cycloid comes from one rotation of the circle and so is described by $0 \le \theta \le 2\pi$.

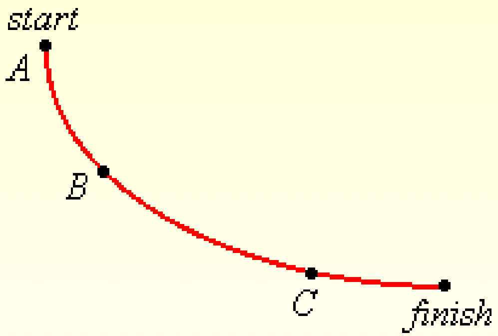
Some properties of cycloid:

(1) A particle slides along the curve from point *A* to a lower point *B* not directly beneath *A*. Among all possible curves joining *A* to *B*, the particle will take the least time if the curve is an inverted arch of a cycloid.

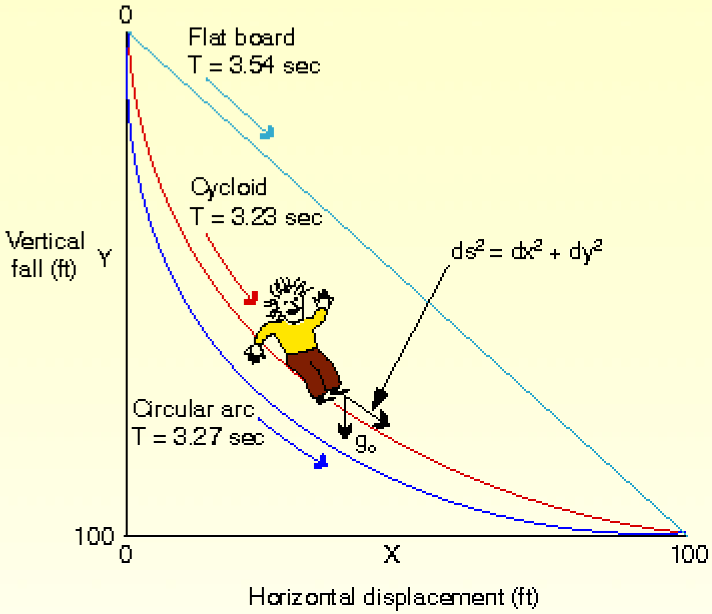


If we take one-half of a cycloid and turn it upside-down we get the brachistochrone for bead.





(2) Notice that it is vertical at the start to get up lots of initial speed and then flattens out at the end. This path has another interesting property, namely if the ride could somehow be started from rest at point *B* or *C* , the ride would last the same length of time as if it started from point *A* .


0
Flat board
T = 3.54 sec
Cycloid
T = 3.23 sec
$ds^2 = dx^2 + dy^2$
Vertical
fall (ft)
Y
Circular arc
T = 3.27 sec
g_o
100
0
X
100
Horizontal displacement (ft)

Thanks for your attention