Differential Equations

Lecture 24

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Review of Previous Lecture:

The Laplace transformations is the special case of * of a function f(x) is defined as:

$$L\left[f(x)\right]=\int_{0}^{\infty} e^{-px} f(x)dx = F(p)$$

List of Laplace Transforms

	f(x)	L(f)		f(x)	L(f)
1	1	1/p	7	cos ax	$\frac{p}{p^2+a^2}$
2	x	$1/p^2$	8	sin ax	$\frac{a}{p^2+a^2}$
3	x^2	$2!/p^3$	9	cosh ax	$\frac{p}{p^2-a^2}$
4	x^n (n=0, 1,…)	$\frac{n!}{p^{n+1}}$	10	sinh ax	$\frac{a}{p^2-a^2}$
5	e^{ax}	$\frac{1}{p-a}$	11	$e^{ax}\cos \omega x$	$\frac{p-a}{(p-a)^2+\omega^2}$
6	$e^{ax}x^n$	$\frac{n!}{(p-a)^{n+1}}$	12	$e^{ax}\sin \omega x$	$\frac{\omega}{(p-a)^2+\omega^2}$

Inverse Laplace transformation

When $f(x)$ is continuous, we have:

$$F(p) = L(f(x))$$

$$f(x) = L^{-1}(F(p))$$

It is customary to call L^{-1} the inverse laplace transformation, and to refer to $f(x)$ as the inverse Laplace transform of $F(p)$.

Since L is linear, it is evident that L^{-1} is also linear.

$$L^{-1}\left(\frac{1}{p^2+p}\right)=L^{-1}\left(\frac{1}{p(p+1)}\right)=L^{-1}\left(\frac{1}{p}+\frac{-1}{p+1}\right)$$

$$=L^{-1}\left(\frac{1}{p}\right)-L^{-1}\left(\frac{1}{p-(-1)}\right)=1-e^{-x}$$

$$L^{-1}\left(\frac{1}{p^4+p^2}\right)=L^{-1}\left(\frac{1}{p^2(p^2+1)}\right)=L^{-1}\left(\frac{1}{p^2}+\frac{-1}{p^2+1}\right)$$

$$=L^{-1}\left(\frac{1}{p^2}\right)-L^{-1}\left(\frac{1}{p^2+1}\right)=x-\sin x$$

$$L^{-1}\left(\frac{1}{p^2+4p+5}\right)=L^{-1}\left(\frac{1}{(p-2)^2+1^2}\right)=e^{2x}\sin x$$

$$L^{-1}\left(\frac{6}{(p+2)+9}\right)=2L^{-1}\left(\frac{3}{(p+2)^2+3^2}\right)$$

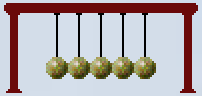
$$=2e^{-2x}\sin 3x$$

$$L^{-1}\left(\frac{12}{(p+3)^4}\right)=2L^{-1}\left(\frac{3!}{(p+3)^4}\right)=2e^{-3x}x^3$$

$$L^{-1}\left(\frac{p+3}{p^2+2p+5}\right)=L^{-1}\left(\frac{p+1+2}{(p+1)^2+2^2}\right)$$

$$=L^{-1}\left(\frac{p+1}{(p+1)^2+2^2}\right)+L^{-1}\left(\frac{2}{(p+1)^2+2^2}\right)$$

$$=e^{-x}\cos 2x+e^{-x}\sin 2x$$



Transforms of Derivatives

$$L(y') = pL(y) - y(0)$$

$$L(y') = \int_0^\infty e^{-px} y' dx = \lim_{b \to +\infty} \int_0^b e^{-px} y' dx$$

$$y' dx = dv \quad \to \quad y = v$$

$$e^{-px} = u \quad \to \quad -pe^{-px} dx = du$$

$$L(y') = \lim_{b \to +\infty} \left(ye^{-px} \Big|_0^b + \int_0^b pe^{-px} y dx \right)$$

$$= \lim_{b \to +\infty} \left(y(b)e^{-pb} - y(0)e^0 + p\int_0^b e^{-px} y dx \right)$$

$$L(f') = -y(0) + pL(y) = pL(y) - y(0) \qquad p > 0$$

$$L(y'') = p^2 L(y) - py(0) - y'(0)$$

$$L(y'') = L((y'))' \quad = pL(y') - y'(0)$$

$$= p(pL(y) - y(0)) - y'(0) =$$

$$= p^2 L(y) - py(0) - y'(0)$$

Differential Equation Solution using Laplace transformation

$$y' + y = e^x \qquad y(0) = 1$$

$$L(y' + y) = L(e^x) \quad \Rightarrow L(y') + L(y) = L(e^x)$$

$$pL(y) - y(0) + L(y) = L(e^x)$$

$$pL(y) - 1 + L(y) = \frac{1}{p-1}$$

$$(p+1)L(y) = \frac{1}{p-1} + 1 = \frac{1 + p - 1}{p-1} = \frac{p}{p-1}$$

$$L(y) = \frac{p}{(p-1)(p+1)}$$

$$y = L^{-1}\left(\frac{p}{(p-1)(p+1)}\right) = L^{-1}\left(\frac{1/2}{p-1} + \frac{1/2}{p+1}\right)$$

$$y = \frac{1}{2}L^{-1}\left(\frac{1}{p-1}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{p+1}\right)$$

$$y = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$$

$$y' + 2y = e^{-x} \qquad y(0) = 2$$

$$L(y' + 2y) = L(e^{-x}) \Rightarrow L(y') + 2L(y) = L(e^{-x})$$

$$pL(y) - y(0) + 2L(y) = L(e^{-x})$$

$$(p+2)L(y) = \frac{1}{p+1} + 2 = \frac{2p+3}{p+1}$$

$$L(y) = \frac{2p+3}{(p+1)(p+2)}$$

$$y = L^{-1}\left(\frac{2p+3}{(p+1)(p+2)}\right)$$

$$y = L^{-1}\left(\frac{1}{p+1} + \frac{1}{p+2}\right)$$

$$y = L^{-1}\left(\frac{1}{p+1}\right) + L^{-1}\left(\frac{1}{p+2}\right)$$

$$y = e^{-x} + e^{-2x}$$

$$y'' + 4y = 4x \quad y'(0) = 5 \quad y(0) = 1$$

$$L(y'' + 4y) = L(4x)$$

$$L(y'') + 4L(y) = 4L(x)$$

$$p^2 L(y) - py(0) - y'(0) + 4L(y) = 4L(x)$$

$$(p^2 + 4)L(y) = \frac{4}{p^2} + p + 5 = \frac{4 + p^3 + 5p^2}{p^2}$$

$$L(y) = \frac{p^3 + 5p^2 + 4}{p^2(p^2 + 4)} \qquad y = L^{-1}\left(\frac{p^3 + 5p^2 + 4}{p^2(p^2 + 4)}\right)$$

$$y = L^{-1}\left(\frac{p^3 + 5p^2 + 4}{p^2(p^2 + 4)}\right)$$

$$y = L^{-1}\left(\frac{1}{p^2} + \frac{p + 4}{p^2 + 4}\right) = L^{-1}\left(\frac{1}{p^2} + \frac{p}{p^2 + 4} + \frac{4}{p^2 + 4}\right)$$

$$y = L^{-1}\left(\frac{1}{p^2}\right) + L^{-1}\left(\frac{p}{p^2 + 4}\right) + 2L^{-1}\left(\frac{2}{p^2 + 4}\right)$$

$$y = x + \cos 2x + 2\sin 2x$$

یکی دیگر از خاصیت های تبدیل لاپلاس خاصیت مضرب می باشد .

قضیه :فرض کنید $F(p)=L(f(x))$ آنگاه

$$L(xf(x)) = -\frac{d}{dp}F(p)$$

$$-\frac{d}{dp}F(p) = -\frac{d}{dp}\int_0^\infty e^{-px} f(x)dx = -\int_0^\infty \frac{\partial}{\partial p} e^{-px} f(x)dx$$

$$= -\int_0^\infty (-x)e^{-px} f(x)dx$$

$$= \int_0^\infty e^{-px}(xf(x)dx) = L(xf(x))$$

$$L(x^2 f(x)) = (-1)^2 \frac{d^2}{dp^2} F(p)$$

$$L(x^2 f(x)) = L(x \cdot x f(x))$$

$$= (-1) \frac{d}{dp} L(x f(x))$$

$$= (-1)^2 \frac{d}{dp} \frac{d}{dp} L(f(x)) = (-1)^2 \frac{d^2}{dp^2} F(p)$$

به استقراء نتیجه می شود که

$$L(x^n f(x)) = (-1)^n \frac{d^n}{dp^n} L(f(x))$$

$$L(x\sin x) = -\frac{d}{dp}\left(\frac{1}{p^2+1}\right) = -\frac{-2p}{(p^2+1)^2}$$

$$= \frac{2p}{(p^2+1)^2}$$

$$L(x\cos x) = -\frac{d}{dp}\left(\frac{p}{p^2+1}\right) = -\frac{p^2+1-2p^2}{(p^2+1)^2}$$

$$= \frac{p^2-1}{(p^2+1)^2}$$

$$L(x^2 \sin x) = -\frac{d}{dp}\left(\frac{2p}{(p^2+1)^2}\right)$$

$$= -\frac{2(p^2+1)^2 - 8p^2(p^2+1)}{(p^2+1)^4}$$

$$= \frac{-6p^4 - 4p^2 + 2}{(p^2+1)^4}$$

$$f(x) = x^{-1/2}$$

$$L(x^{-1/2}) = \int_0^\infty e^{-px} x^{-1/2} dx$$

$$px = t \qquad L(x^{-1/2}) = p^{-1/2} \int_0^\infty e^{-t} t^{-1/2} dt$$

$$t = s^2 \qquad L(x^{-1/2}) = 2p^{-1/2} \int_0^\infty e^{-s^2} ds$$

$$\int_0^\infty e^{-s^2} ds = \frac{\sqrt{\pi}}{2} \qquad L(x^{-1/2}) = \sqrt{\pi / p}$$

Thanks for your attentions