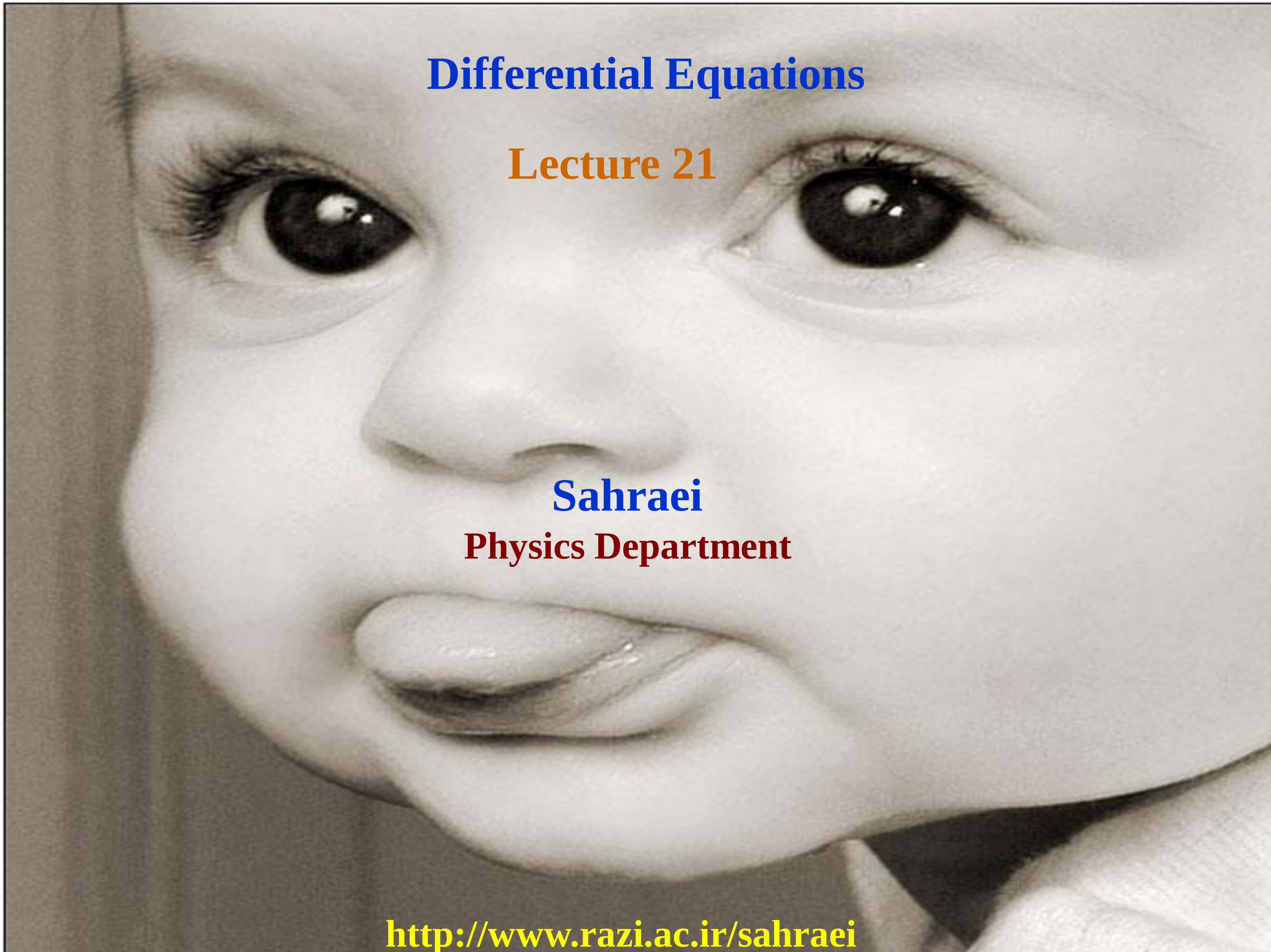

Differential Equations

Lecture 21

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Bessel's Functions, The Gamma Function

$$x^2 y'' + xy' + (x^2 - p^2)y = 0$$

Where p is a non-negative constant

Its solutions are known as Bessel functions

$$y'' + \frac{1}{x}y' + \frac{x^2 - p^2}{x^2}y = 0$$

It is apparent that the origin (x=0) is a regular singular point because

$$xp(x) = 1, \quad x^2 Q(x) = x^2 - p^2$$

are analytic at x=0.

We obtain the **indicial equation** $m^2 - p^2 = 0$

$$m_1 = p, \quad m_2 = -p$$

$$y = x^p \sum_0^{\infty} a_n x^n = \sum_0^{\infty} a_n x^{n+p}$$

$$y' = \sum_0^{\infty} a_n (n+p) x^{n+p-1}$$

$$y'' = \sum_0^{\infty} a_n (n+p-1)(n+p) x^{n+p-2}$$

$$y'' + \frac{1}{x} y' + \frac{x^2 - p^2}{x^2} y = 0$$

$$\sum_{n=0}^{\infty} a_n (n+p-1)(n+p) x^{n+p} + \sum_{n=0}^{\infty} a_n (n+p) x^{n+p}$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+p+2} - p^2 \sum_{n=0}^{\infty} a_n x^{n+p} = 0$$

$$a_n = \frac{-a_{n-2}}{n(n+2p)}$$

We know that a_0 is nonzero and arbitrary.

$$a_1 = a_3 = a_5 = \ldots = 0 \rightarrow a_n = 0, \ \ n = 1,3,5,\ldots$$

$$a_0 \ , \qquad a_2 = -\frac{a_0}{2(2p+2)} \ ,$$

$$a_4 = -\frac{a_2}{4(2p+4)} = \frac{a_0}{2\times 4(2p+2)(2p+4)}$$

$$a_6 = -\frac{a_4}{6(2p+6)} = -\frac{a_0}{2\times 4\times 6(2p+2)(2p+4)(2p+6)},\ldots$$

$$y = a_0 x^p \left[1 - \frac{x^2}{2^2 (p+1)} + \frac{x^4}{2^4 2! (p+1)(p+2)} \right.$$

$$\left. - \frac{x^6}{2^6 3! (p+1)(p+2)(p+3)} + \ldots \right]$$

$$= a_0 x^p \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n} n! (p+1) \ldots (p+n)}$$

The *Bessel function of the first kind of order p*, denoted by $J_p(x)$, is defined by putting $a_0 = \frac{1}{2^p p!}$

$$J_p(x) = \frac{x^p}{2^p p!} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n} n! (p+1) \ldots (p+n)}$$

$$J_p(x) = \frac{x^p}{2^p p!} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n} n!(p+1)\dots(p+n)}$$

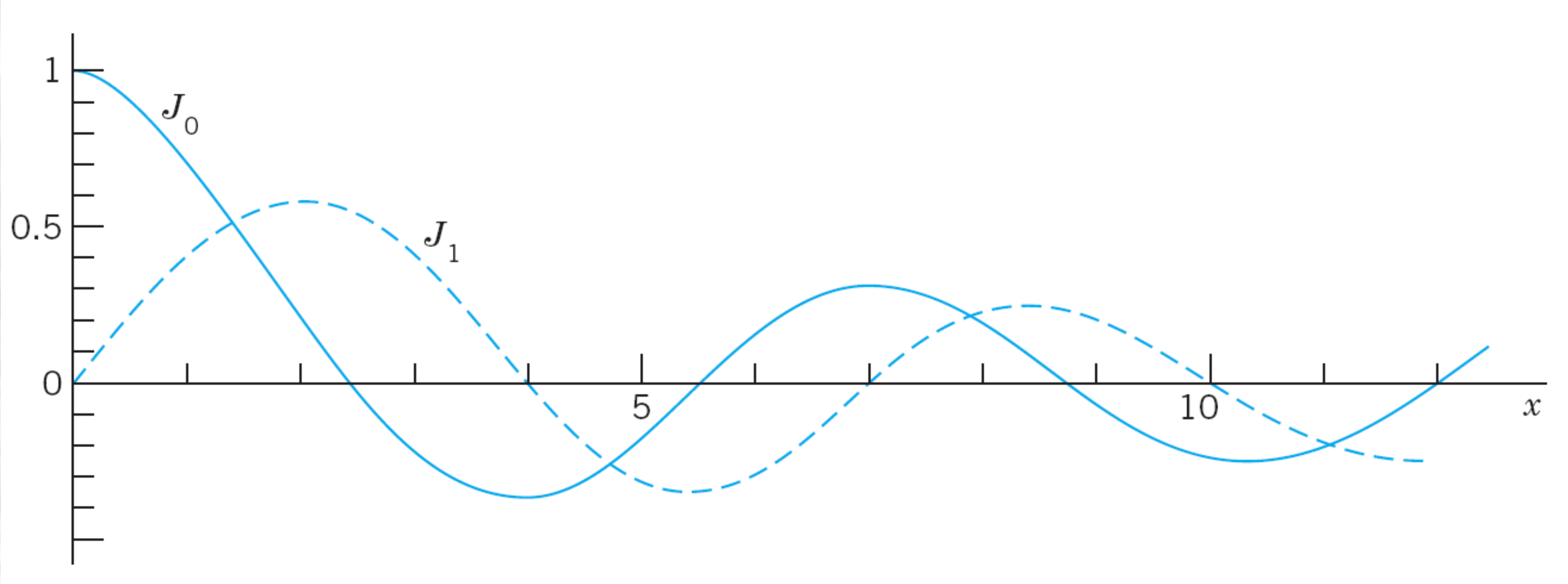
$$= \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n+p}}{n!(p+n)!}$$

The most useful Bessel functions are those of order 0 and 1, which are:

$$J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} \left(\frac{x}{2}\right)^{2n} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \times 4^2} - \frac{x^6}{2^2 \times 4^2 \times 6^2} + \dots$$

$$J_1(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)(n+1)!} \left(\frac{x}{2}\right)^{2n+1} = \frac{x}{2} - \frac{1}{1!2!} \left(\frac{x}{2}\right)^3 + \frac{1}{2!3!} \left(\frac{x}{2}\right)^5 - \dots$$

Bessel functions of the first kind J_0 and J_1



These graphs display several interesting properties of the functions $J_0(x)$ and $J_1(x)$: each has a damped oscillatory behavior producing an infinite number of positive zeros; and these zeros occur alternately, in a manner suggesting the function *cos x* and *sin x*. this loose analogy is strengthened by the relation $J_0'(x) = -J_1(x)$

The Gamma Function

For $p > 0$, **the *gamma function* is defined by**

$$\Gamma(p) = \int_0^{\infty} t^{p-1} e^{-t} dt \quad , \quad p > 0$$

تعریف می شود، لازم است.این انتگرال به ازای همه p های بزرگتر از صفر همگراست.

$$\Gamma(p+1) = p\Gamma(p)$$

$$\Gamma(p+1) = \int_0^{\infty} t^p e^{-t} dt$$

$$\Gamma(p+1) = \lim_{b\to\infty} \int_0^b t^p e^{-t} dt = \lim_{b\to\infty} \left[-t^p e^{-t} \Big|_0^b \right] + p\int_0^{\infty} t^{p-1} e^{-t} dt$$

$$= p(\lim_{b\to\infty} \int_0^b t^{p-1} e^{-t} dt) = p\Gamma(p)$$

$$\sin ce \;\; b^p / e^b \to 0 \;\; as \;\; b \to \infty$$

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt \;= -\int_0^\infty t^{p-1}\, de^{-t}$$

$$= -\{[t^{p-1}\; e^{-t}]_0^\infty \;-\; (p-1)\int_0^\infty t^{p-2}\; e^{-t}\; dt\}$$

$$= (p-1)\int_0^\infty t^{p-2}\; e^{-t}\; dt = (p-1)\; \Gamma(p-1)$$

$$\Gamma(p) = (p-1)\; \Gamma(p-1)$$

$$\Gamma(p+1) = p\Gamma(p)$$

$$P = 1 \rightarrow \Gamma(1) = \int_0^\infty e^{-t} dt = 1 \qquad \Gamma(2) = 1\Gamma(1) = 1$$

$$\Gamma(3) = 2\Gamma(2) = 2\times 1 = 2$$

$$\Gamma(4) = 3\Gamma(3) = 3\times 2 = 3\times 2\times 1 = 3\,!$$

In general for any integer $n \geq 0$ $\quad \Gamma(n+1) = n!$

Example (1): Find the value of 0! $\quad 0! = \Gamma(1) = 1$

Example (2): Find the value of (3.1)!

$$(3.1)! = \Gamma(3.1+1) = (3.1)\Gamma(3.1) = (3.1)(2.1)\Gamma(2.1)$$

$$= (3.1)(2.1)\Gamma(2.1) \quad = (3.1)(2.1)(1.1)\Gamma(1.1)$$

$$\Gamma(1.1) = 0.951351 \qquad (3.1)! = 6.81262$$

Example (3): $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt \qquad t = x^2 \qquad dt = 2x\,dx$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{-1} e^{-x^2} 2x\,dx = 2\int_0^\infty e^{-x^2} dx$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Example (4): $\Gamma(3/2) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$

Example (5): $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$

Example (6): $\Gamma\left(3\frac{1}{2}\right)$

$$\Gamma(p) = (p-1)\Gamma(p-1)$$

$$\Gamma\left(3\frac{1}{2}\right) = \frac{5}{2}\Gamma\left(2\frac{1}{2}\right) = \frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi} = \frac{15}{8}\sqrt{\pi}$$

$$\Gamma(p+1) = p\Gamma(p) \quad \Rightarrow \quad \Gamma(p) = \frac{\Gamma(p+1)}{p}$$

$$-1 < p < 0 \quad \rightarrow 0 < p+1 < 1$$

$$-2 < p < -1 \quad \rightarrow -1 < p+1 < 0$$

$$\lim_{p \to 0} \Gamma(p) = \lim_{p \to 0} \frac{\Gamma(p+1)}{p} = \pm\infty$$

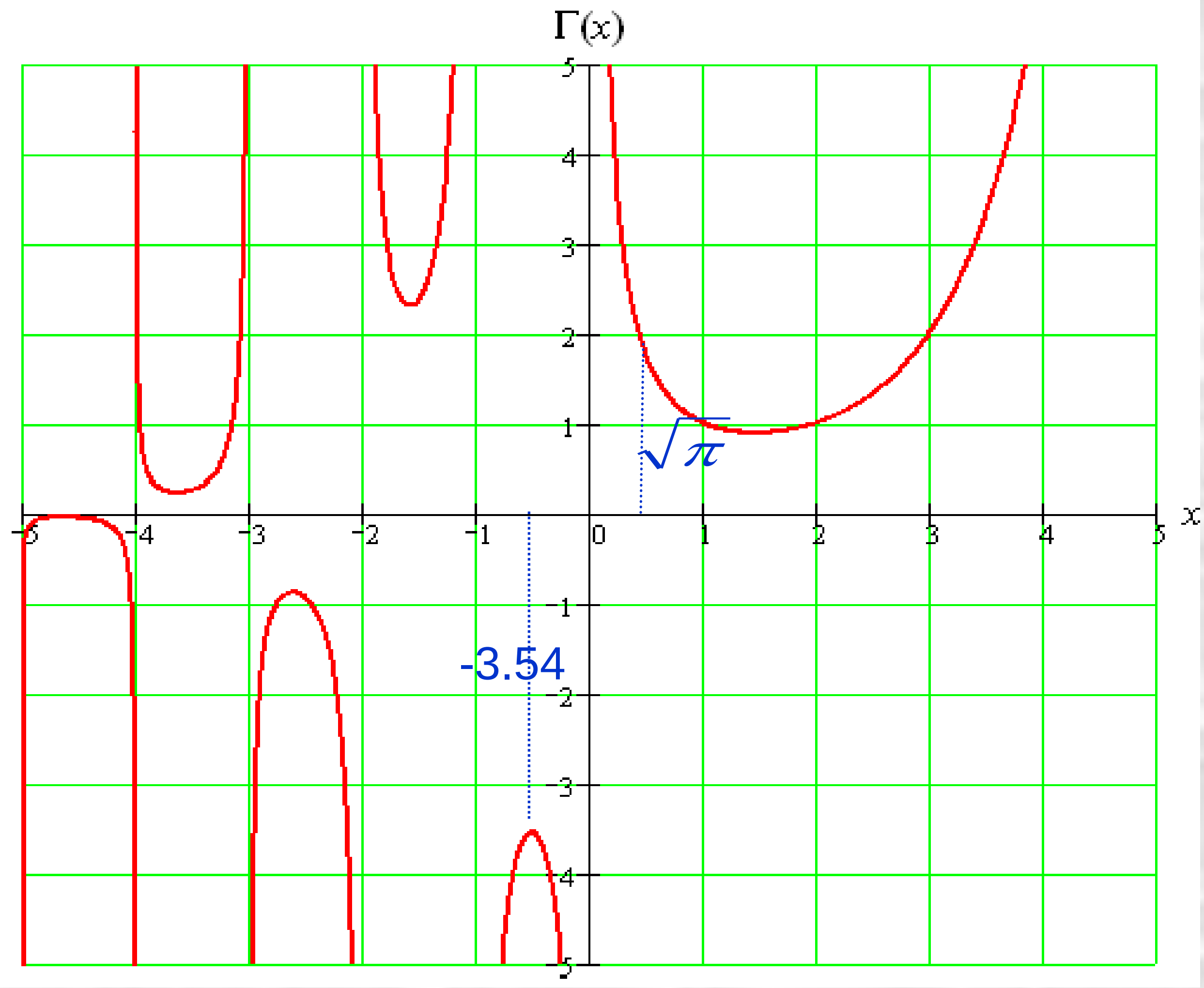
$\Gamma(p) \rightarrow \infty$ **when p is a negative integer or $p = 0$**

Example

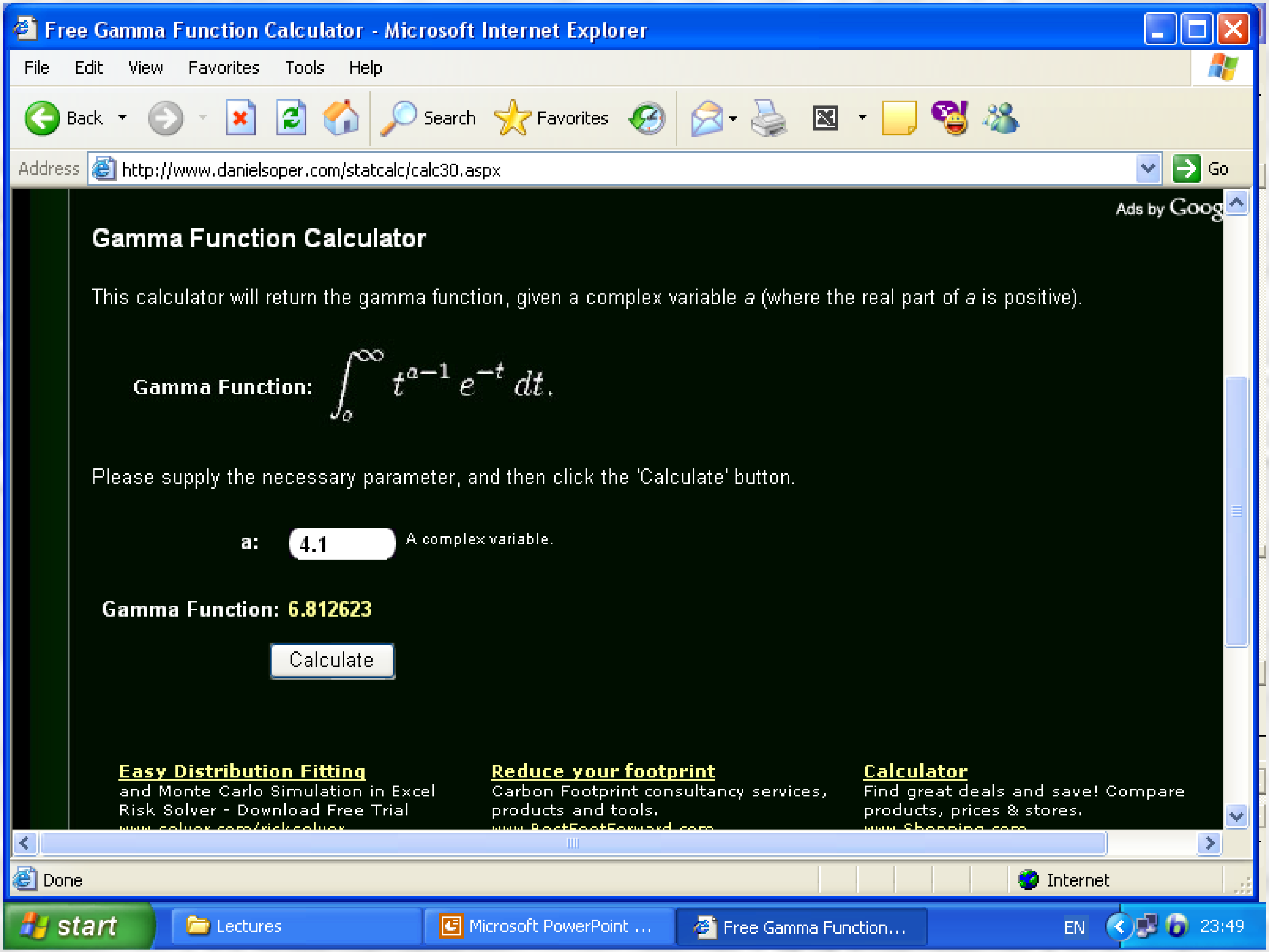
$$\Gamma(p) = \frac{\Gamma(p+1)}{p} \qquad -1 < p < 0$$

$$\Gamma(-\frac{1}{2}) = \frac{\Gamma(\frac{1}{2})}{-\frac{1}{2}} = -2\Gamma(\frac{1}{2}) = -2\sqrt{\pi}$$

$$\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$$



http://www.danielsoper.com/statcalc/calc30.aspx



The general solution of Bessel's equation

$$x^2 y'' + xy' + (x^2 - p^2)y = 0$$

Case 1 -Where p is a non-negative constant

$$y = x^p \sum_0^\infty a_n x^n = \sum_0^\infty a_n x^{n+p}$$

$$m^2 - p^2 = 0$$

$$m_1 = p \rightarrow J_p(x) = \sum_{n=0}^\infty (-1)^n \frac{1}{(n!)(n+p)!} \left(\frac{x}{2}\right)^{2n+p}$$

$$J_p(x) = \sum_{n=0}^\infty (-1)^n \frac{1}{\Gamma(n+1)\Gamma(n+p+1)!} \left(\frac{x}{2}\right)^{2n+p}$$

$$m_2 = -p \qquad m_1 - m_2 = 2p$$

$$a_n = \frac{-a_{n-2}}{n(n-2p)} \qquad a_{n+2} = \frac{-a_n}{(n+2)(n+2-2p)}$$

The only exception is that if p=1/2, then letting n=1 we see that there is no compulsion to choose a_1=0. However, since all we want is a particular solution, it is certainly permissible to put a_1=0. The same problem arises when p=3/2 ant n=3, and so on.

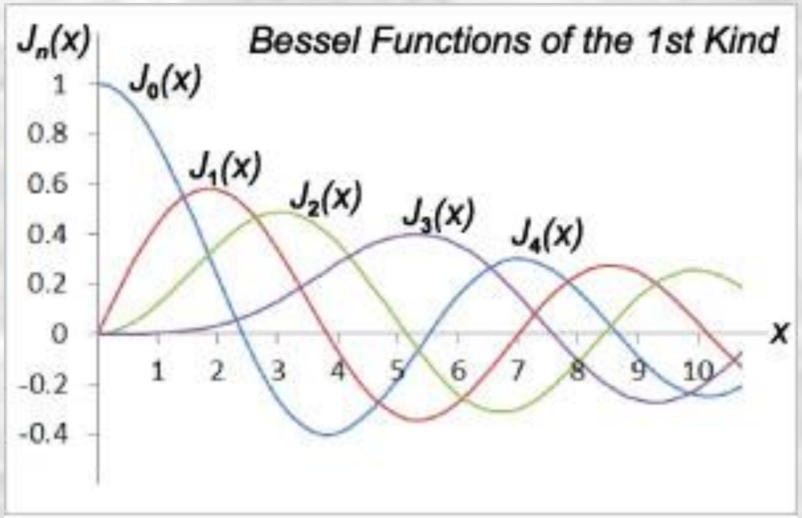
$$a_1 = a_3 = \ldots = 0$$

$$J_{-p}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n-p}}{n!(-p+n)!}$$

$$J_{-p}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n-p}}{\Gamma(n+1)\Gamma(-p+n+1)!}$$

The Bessel functions can be calculated in most mathematical software packages

For example, the Bessel functions of the 1st kind of orders $p = 0$ to $p = 4$ are shown in Figure



$$n = 0 \rightarrow \frac{1}{(-p)!}(x/2)^{-p}$$

So $J_{-p}(x)$ is unbounded near x=0. Since $J_p(x)$ is bounded near x=0, these two solution are independent and

$$y = c_1 J_p(x) + c_2 J_{-p}(x)$$

***P* not an integer**

Case 2: The situation is entirely different when *p* is an integer

$$m \geq 0$$

If the order ν of the Bessel differential equation is an integer, the Bessel functions $J_\nu(x)$ and $J_{-\nu}(x)$ can become dependent from each other. In this case the general solution is described by another formula:

$$J_{-m}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n-m}}{n!(-m+n)!}$$

$$= \sum_{n=m}^{\infty} (-1)^n \frac{(x/2)^{2n-m}}{n!(-m+n)!}$$

$$\frac{1}{(-m+n)!} = 0 \qquad when \quad n = 0,1,\ldots,m-1$$

$$n \to n+m \qquad \text{summation } at \quad \text{n} = 0$$

$$J_{-m}(x) = \sum_{n=0}^{\infty} (-1)^{n+m} \frac{(x/2)^{2(n+m)-m}}{(n+m)!\,n!}$$

$$J_{-m}(x) = \sum_{n=0}^{\infty} (-1)^{n+m} \frac{(x/2)^{2(n+m)-m}}{(n+m)!\,n!}$$

$$= (-1)^m \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n+m}}{n!(m+n)!} = (-1)^m J_m(x)$$

This show that $J_{-m}(x)$ is not independent of $J_m(x)$, so in this case:

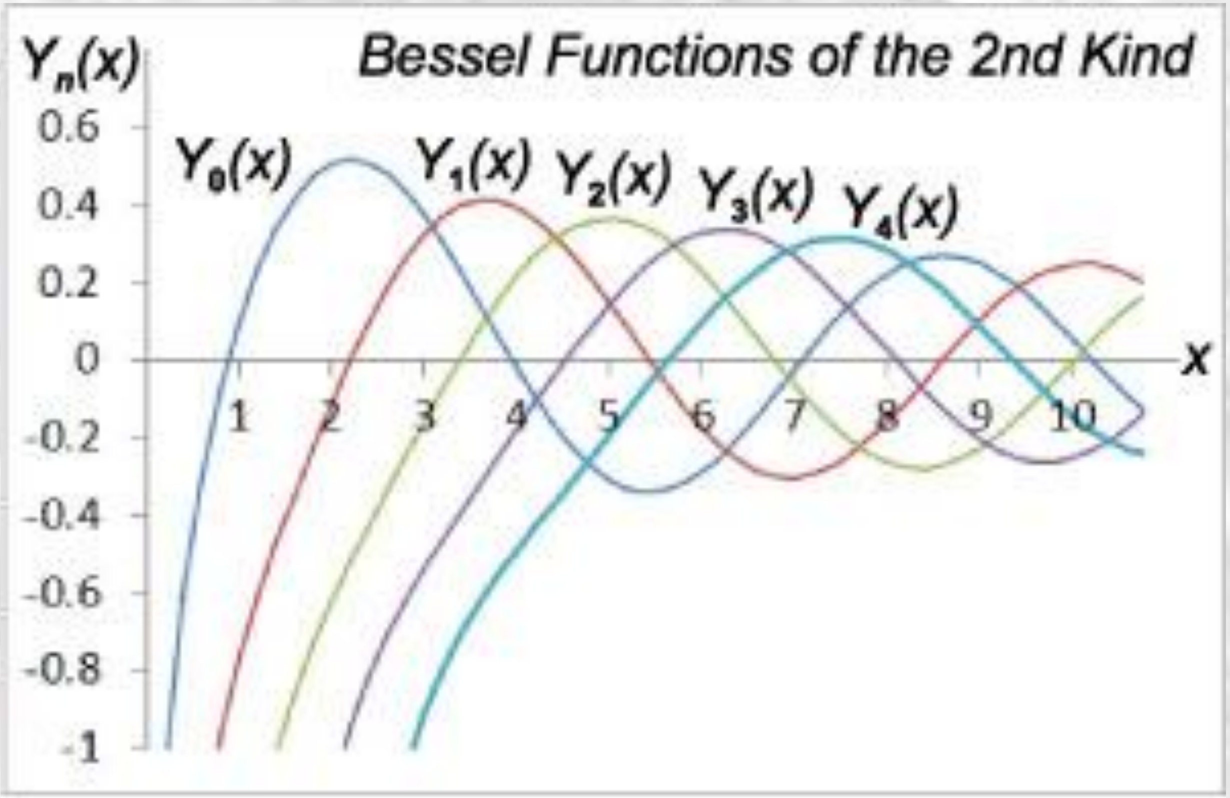
$$y = c_1 J_m(x) + c_2 J_{-m}(x)$$

is not the general solution of Bessel equation and the search continues.

The *Bessel function of the second kind* $Y_p(x)$ can be expressed through the Bessel functions of the first kind $J_p(x)$ and $J_{-p}(x)$:

$$Y_p(x) = \frac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin\ p\pi}$$

The standard Bessel function of the 2nd kind

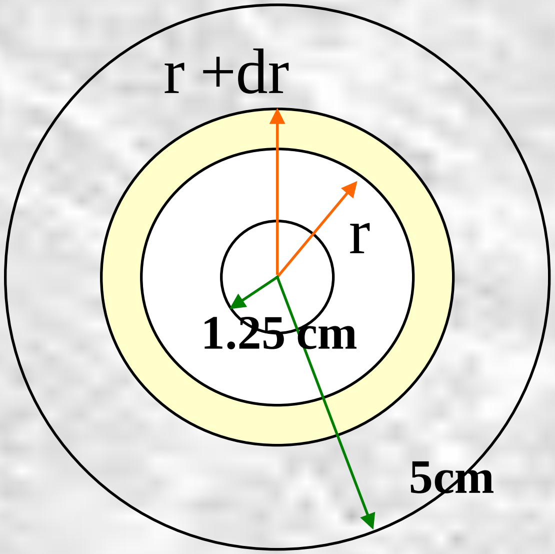


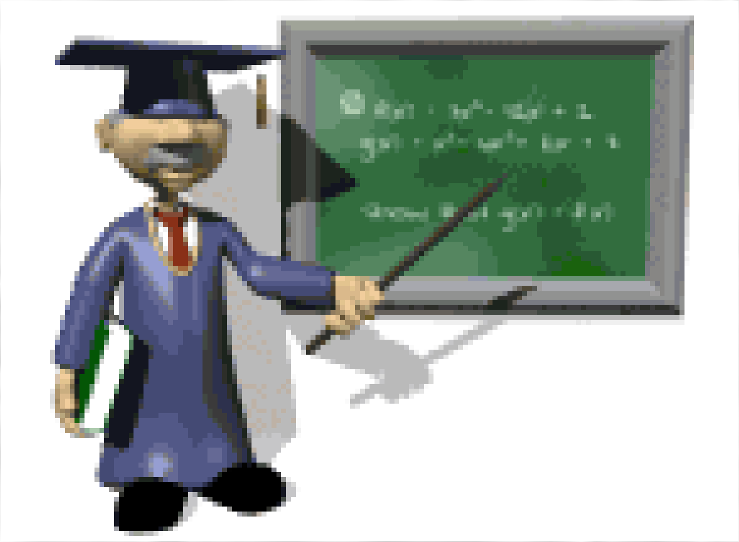
Note: Actually the general solution of the differential equation expressed through Bessel functions of the first and second kind is valid for non-integer orders as well.

$$y(x) = c_1 J_p(x) + c_2 Y_p(x) \qquad p \text{ not an integer.}$$

Bessel's Equation Example

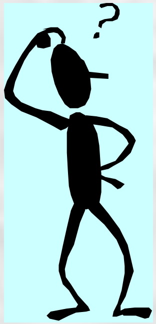
Two thin wall metal pipes of 2.5 cm external diameter and joined by flanges 1.25 cm thick and 10 cm diameter, are carrying steam at 120 °C. If the conductivity of the flange metal *k*=400 W/m °C and the exposed surfaces of the flanges lose heat to the surrounding at T_1=15 °C according to a heat transfer coefficient *h*=12 W/m^2 °C, find the rate of heat loss from the pipe, and the proportion which leaves the rim of the flange.





Thanks for your attentions

Find singular points and kind of singularity



$$y'' + \frac{4}{(1-x)^2} y' + \frac{12}{1-x} y = 0$$

$$1 - x = 0 \Rightarrow x = 1 \Rightarrow \lim_{x \to 1} p(x) \to \infty \quad \Rightarrow x = 1 \text{ S.P.}$$

$$\lim_{x \to 1} Q(x) \to \infty$$

$$(x - x_0) p(x) \quad \Rightarrow (x-1) \frac{4}{(1-x)^2} = \frac{4}{\underset{x \to 1}{1-x}}$$

$$\Rightarrow (x - x_0) p(x) \to \infty$$ **irregular singular point**