Differential Equations

Lecture 20

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Review of Previous Lecture:

$$y'' + P(x)y' + Q(x)y = 0$$

P(x) **and** ***Q(x)*** **are analytic at the x_0 and therefore have power series expansions. In these cases x_0 is called ordinary point of equation.**

$$P(x) = \sum_{n=0}^{\infty} p_n x^n \qquad Q(x) = \sum_{n=0}^{\infty} q_n x^n$$

Equation also is analytic

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad x_0 = 0 \rightarrow y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

A singular point x_0 of equation * is said to be regular if the function

$(x-x_0)P(x)$ $\qquad$ $(x-x_0)^2Q(x)$ **are analytic, and irregular otherwise.**

$$y = x^m \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad x_0 = 0 \rightarrow y = x^m \sum_{n=0}^{\infty} a_n x^n$$

$$f(m) = 0 \quad \rightarrow m_1, m_2, \qquad m_1 > m_2$$

1. If the two roots differ by a noninteger, two solutions can be obtained.

$$m_1 - m_2 \neq N \rightarrow \begin{cases} y_1 = x^{m_1} \sum_{n=0}^{\infty} a_n x^n \\ y_2 = x^{m_2} \sum_{n=0}^{\infty} a_n x^n \end{cases}$$

2. If the two roots differ by an integer, the larger will yield a solution. The smaller may or may not.

$$m_1 - m_2 = N \rightarrow \begin{cases} y_1 = x^{m_1} \sum_{n=0}^{\infty} a_n x^n \\ y_2 = k y_1 \ln x + x^{m_2} \sum_{n=0}^{\infty} c_n x^n \end{cases}$$

3. If the two roots are equal, only one solution can be obtained.

$$m_1 = m_2 \rightarrow \begin{cases} y_1 = x^{m_1} \sum_{n=0}^{\infty} a_n x^n \\ y_2 = y_1 \ln x + x \sum_{n=1}^{\infty} c_n x^n \end{cases}$$

Example $$2x^2y'' + x(2x+1)y' - y = 0$$

$$y'' + \frac{x}{2x^2}(2x+1)y' - \frac{1}{2x^2}y = 0$$

$$y'' + \frac{1/2 + x}{x}y' + \frac{-1/2}{x^2}y = 0$$

$$xP(x) = 1/2 + x \qquad x^2Q(x) = -1/2$$

So x=0 is a regular singular point. Oue assumed Frobenius series solution:

$$y = \sum_{n=0}^{\infty} a_n x^{m+n} \qquad y'(x) = \sum_{n=0}^{\infty} a_n (m+n) x^{m+n-1}$$

$$y''(x)=\sum_{n=0}^{\infty} a_n(m+n)(m+n-1)x^{m+n-2}$$

$$2x^2y''+x(2x+1)y'-y=0$$

$$2\sum_{n=0}^{\infty} a_n(m+n)(m+n-1)x^{m+n}+2\sum_{n=0}^{\infty}(m+n)a_n x^{m+n+1}$$

$$+\sum_{n=0}^{\infty}(m+n)a_n x^{m+n}-\sum_{n=0}^{\infty}a_n x^{m+n}=0$$

$$x^m\left(\sum_{n=0}^{\infty}2a_n(m+n)(m+n-1)x^n+\sum_{n=0}^{\infty}2(m+n)a_n x^{n+1}\right.$$

$$\left.+\sum_{n=0}^{\infty}(m+n)a_n x^n-\sum_{n=0}^{\infty}a_n x^n\right)=0$$

$$n=0\rightarrow 2m(m-1)a_0+ma_0-a_0=0$$

$$2m(m-1)a_0 + ma_0 - a_0 = 0$$

$$\left[(2m+1)(m-1)\right]a_0 = 0$$

$$(2m+1)(m-1) = 0 \Rightarrow \begin{cases} m_1 = 1 \\ m_2 = -1/2 \end{cases}$$

$$m_1 - m_2 = 1 - (-1/2) = 3/2$$

$$\sum_{n=0}^{\infty} 2a_n(n+1)nx^n + \sum_{n=1}^{\infty} 2na_{n-1}x^n + \sum_{n=0}^{\infty}(n+1)a_n x^n$$

$$-\sum_{n=0}^{\infty} a_n x^n = 0$$

$$2(n+1)na_n + 2na_{n-1} + (n+1)a_n - a_n = 0$$

$$2(n+1)na_n + 2na_{n-1} + na_n = 0$$

$$na_n(2n+3) + 2na_{n-1} = 0$$

$$a_n = -\frac{2n}{n(2n+3)} a_{n-1}$$

$$n=1 \rightarrow a_1 = -\frac{2}{5} a_0 \quad n=2 \rightarrow a_2 = -\frac{4}{14} a_1 = \frac{4}{35} a_0$$

$$n=3 \rightarrow a_3 = -\frac{6}{27} a_2 = -\frac{8}{7\times9\times5} a_0$$

$$n=4 \rightarrow a_4 = -\frac{8}{4\times11} a_3 = \frac{16}{7\times9\times5\times11} a_0$$

$$y=\sum_{n=0}^{\infty} a_n x^{m+n}$$

$$y_1 = x a_0 (1-\frac{2}{5}x+\frac{4}{35}x^2-\frac{8}{360}x^3+\ldots)$$

$$y_1 = x(1-\frac{2}{5}x+\frac{4}{35}x^2-\frac{8}{360}x^3+\ldots)$$

$$x^m(\sum_{n=0}^{\infty} 2a_n(m+n)(m+n-1)x^n+\sum_{n=0}^{\infty} 2(m+n)a_n x^{n+1}$$

$$+\sum_{n=0}^{\infty}(m+n)a_n x^n-\sum_{n=0}^{\infty} a_n x^n)=0$$

$$[2(-1/2+n)(-3/2+n)+(-3/2+n)]a_n+2(-3/2+n)a_{n-1}=0$$

$$a_n = -\frac{(n-3/2)}{(n-3/2)n} a_{n-1}$$

$$n=1 \rightarrow a_1 = \frac{1/2}{-1/2} a_0 = -a_0 \qquad n=2 \rightarrow a_2 = -\frac{1}{2} a_1 = \frac{1}{2} a_0$$

$$n=3 \rightarrow a_3 = -\frac{1}{3} a_2 = -\frac{1}{6} a_0$$

$$y_2(x) = a_0 x^{-1/2}\left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots\right)$$

$$y_2(x) = x^{-1/2}\left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots\right)$$

$$y(x) = c_1 x\left(1 - \frac{2}{5}x + \frac{4}{35}x^2 + \ldots\right) + c_2 x^{-1/2}\left(1 - x + \frac{1}{2}x^2 + \ldots\right)$$

Example

$$2xy''-(3+2x)y'+y=0$$

$$y=x^m\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}a_nx^{n+m} \qquad y'=\sum_{n=0}^{\infty}(n+m)a_nx^{n+m-1}$$

$$y''=\sum_{n=0}^{\infty}(n+m-1)(n+m)a_nx^{n+m-2}$$

$$2x\sum_{n=0}^{\infty}(n+m-1)(n+m)a_nx^{n+m-2}$$

$$-(3+2x)\sum_{n=0}^{\infty}(n+m)a_nx^{n+m-1}+\sum_{n=0}^{\infty}a_nx^{n+m}=0$$

$$\sum_{n=0}^{\infty} 2(n+m-1)(n+m)a_n x^{n+m-1}$$

$$-\sum_{n=0}^{\infty} 3(n+m)a_n x^{n+m-1} - \sum_{n=0}^{\infty} 2(n+m)a_n x^{n+m} + \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

First two series have the lowest "degree" terms, x^{m-1} for n=0.

$$2(m-1)ma_0 - 3ma_0 = 0, \quad (2m^2 - 2m - 3m)a_0 = 0$$

$$2m^2 - 5m = m(2m-5) = 0 \quad m_1 = 0, \quad m_2 = 5/2$$

Substitute m=0 to find one of the two solution.

$$\sum_{n=0}^{\infty} 2(n-1)na_n x^{n-1} - \sum_{n=0}^{\infty} 3na_n x^{n-1} - \sum_{n=0}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} 2(n-1)na_n x^{n-1} - \sum_{n=1}^{\infty} 3na_n x^{n-1} - \sum_{n=0}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Re-index the first two series to begin with n=0:

$$\sum_{n=0}^{\infty} 2n(n+1)a_{n+1} x^n - \sum_{n=0}^{\infty} 3(n+1)a_{n+1} x^n - \sum_{n=0}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[2n(n+1)a_{n+1} - 3(n+1)a_{n+1} - 2na_n + a_n\right] x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(2n-3)a_{n+1} - (2n-1)a_n\right] x^n = 0$$

Set all coefficients equal to zero and solve for a_{n+1}:

$$a_{n+1} = \frac{(2n-1)}{(n+1)(2n-3)} a_n, \quad n = 0,1,2,\ldots$$

$$n = 0: \quad a_1 = \frac{1}{1\times 3} a_0$$

$$n = 1: \quad a_2 = -\frac{1}{2\times 1} a_1 = -\frac{1}{1\times 2\times 3\times 1} a_0$$

$$n = 2: \quad a_3 = \frac{3}{3\times 1} a_2 = -\frac{1\times 3}{1\times 2\times 3\times 3\times 1\times 1} a_0$$

$$n = 3: \quad a_4 = \frac{5}{4\times 3} a_3 = -\frac{1\times 3\times 5}{(1\times 2\times 3\times 4)(3\times 1\times 1\times 3)} a_0$$

$$a_{n+1}=\frac{(2n-1)}{(n+1)(2n-3)}a_n, \quad n=0,1,2,\ldots$$

$$n=4: a_5=\frac{7}{5\times 5}a_4=-\frac{1\times 3\times 5\times 7}{(1\times 2\times 3\times 4\times 5)(3\times 1\times 1\times 3\times 5)}a_0$$

$$n=5: a_6=\frac{9}{6\times 7}a_5=-\frac{1\times 3\times 5\times 7\times 9}{(1\times 2\times 3\times 4\times 5\times 6)(3\times 1\times 1\times 3\times 5\times 7)}a_0$$

$$a_1=\frac{1}{1!\times 3}a_0, \quad a_2=-\frac{1}{2!\times 3}a_0, \quad a_3=-\frac{3}{3!\times 3}a_0,$$

$$a_4=-\frac{5}{4!\times 3}a_0, \quad a_5=-\frac{7}{5!\times 3}a_0, \ldots$$

$$y = x^0(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots)$$

$$y = a_0 + \frac{1}{1!\times 3} a_0 x - \frac{1}{2!\times 3} a_0 x^2 - \frac{3}{3!\times 3} a_0 x^3 - \frac{5}{4!\times 3} a_0 x^4 - \frac{7}{5!\times 3} a_0 x^5 - \ldots$$

$$y_1 = 1 + \frac{1}{1!\times 3} x - \frac{1}{2!\times 3} x^2 - \frac{3}{3!\times 3} x^3 - \frac{5}{4!\times 3} x^4 - \frac{7}{5!\times 3} x^5 - \ldots$$

Substitute m=5/2 to find the other solution.

$$\sum_{n=0}^{\infty} 2(n+m-1)(n+m) a_n x^{n+m-1}$$

$$-\sum_{n=0}^{\infty} 3(n+m) a_n x^{n+m-1} - \sum_{n=0}^{\infty} 2(n+m) a_n x^{n+m} + \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

$$\sum_{n=0}^{\infty} 2(n+3/2)(n+5/2)a_n x^{n+3/2} - \sum_{n=0}^{\infty} 3(n+5/2)a_n x^{n+3/2}$$

$$\sum - \sum_{n=0}^{\infty} 2(n+5/2)a_n x^{n+5/2} + \sum_{n=0}^{\infty} a_n x^{n+5/2} = 0$$

$$x^{3/2}(\sum_{n=0}^{\infty} 2\frac{(2n+3)}{2}\frac{(2n+5)}{2}a_n x^n -$$

$$\sum_{n=0}^{\infty} 3\frac{(2n+5)}{2}a_n x^n - \sum_{n=0}^{\infty} 2\frac{(2n+5)}{2}a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1}) = 0$$

Divide by $x^{3/2}$ and multiply by 2:

$$\sum_{n=0}^{\infty}(2n+3)(2n+5)a_n x^n - \sum_{n=0}^{\infty} 3(2n+5)a_n x^n - \sum_{n=0}^{\infty} 2(2n+5)a_n x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

“Kick” the first two terms out from the first two series:

$$15a_0 + \sum_{n=1}^{\infty} 2(2n+3)(2n+5)a_n x^n - 15a_0$$

$$-\sum_{n=1}^{\infty} 3(2n+5)a_n x^n - \sum_{n=0}^{\infty} 2(2n+5)a_n x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

Simplify and re-index first two series:

$$\sum_{n=0}^{\infty} (2(n+1)+3)(2(n+1)+5)a_{n+1} x^{n+1}$$

$$-\sum_{n=0}^{\infty} 3(2(n+1)+5)a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} 2(2n+5)a_n x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty}(2n+5)(2n+7)a_{n+1}x^{n+1}-$$

$$\sum_{n=0}^{\infty}3(2n+7)a_{n+1}x^{n+1}-\sum_{n=0}^{\infty}2(2n+5)a_nx^{n+1}+\sum_{n=0}^{\infty}2a_nx^{n+1}=0$$

$$\sum_{n=0}^{\infty}((2n+5)(2n+7)a_{n+1}-3(2n+7)a_{n+1}-2(2n+5)a_n+2a_n)x^{n+1}=0$$

$$\sum_{n=0}^{\infty}\left((2n+7)(2n+5-3)a_{n+1}-2(2n+5-1)a_n\right)x^{n+1}=0$$

$$\sum_{n=0}^{\infty}\left(2(2n+7)(n+1)a_{n+1}-4(n+2)a_n\right)x^{n+1}=0$$

$$2(2n+7)(n+1)a_{n+1}-4(n+2)a_n=0, \quad n=0,1,2,\ldots$$

$$a_{n+1} = \frac{2(n+2)}{(2n+7)(n+1)} a_n, \quad n = 0,1,2,\ldots$$

$$n = 0: \quad a_1 = \frac{2\times 2}{7\times 1} a_0$$

$$n = 1: \quad a_2 = \frac{2\times 3}{9\times 2} a_1 = \frac{2^2\times 2\times 3}{7\times 9\times 1\times 2} a_0 = \frac{2^2\times 3}{7\times 9} a_0,$$

$$n = 2: \quad a_3 = \frac{2\times 4}{11\times 3} a_2 = \frac{2^3\times 2\times 3\times 4}{7\times 9\times 11\times 1\times 2\times 3} a_0 = \frac{2^3\times 4}{7\times 9\times 11} a_0,$$

$$n = 3: \quad a_4 = \frac{2\times 5}{13\times 4} a_3 = \frac{2^4\times 2\times 3\times 4\times 5}{7\times 9\times 11\times 13\times 1\times 2\times 3\times 4} a_0$$

$$a_4 = \frac{2^4 \times 5}{7\times 9\times 11\times 13} a_0, \quad a_{n+1} = \frac{2(n+2)}{(2n+7)(n+1)} a_n, \quad n = 0,1,2,\ldots$$

$$n = 4: \quad a_5 = \frac{2\times 6}{15\times 5} a_4 = \frac{2^5 \times 2\times 3\times 4\times 5\times 6}{7\times 9\times 11\times 13\times 15\times 1\times 2\times 3\times 4\times 5} a_0$$

$$= \frac{2^5 \times 6}{7\times 9\times 11\times 13\times 15} a_0, \ldots$$

$$y = x^{5/2}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\right)$$

$$y = x^{5/2}(a_0 + \frac{2\times 2}{7} a_0 x + \frac{2^2\times 3}{7\times 9} a_0 x^2 + \frac{2^3\times 4}{7\times 9 \div 11} a_0 x^3$$

$$+ \frac{2^4\times 5}{7\times 9\times 11\times 13} a_0 x^4 + \frac{2^5\times 6}{7\times 9\times 11\times 13\times 15} a_0 x^5 + \ldots)$$

$$y_2 = x^{5/2}(1 + \frac{2\times 2}{7} x + \frac{2^2\times 3}{7\times 9} x^2 + \frac{2^3\times 4}{7\times 9\times 11} x^3$$

$$+ \frac{2^4\cdot 5}{7\times 9\times 11\times 13} x^4 + \frac{2^5\times 6}{7\times 9\times 11\times 13\times 15} x^5 + \ldots)$$

$$y(x) = c_1 y_1 + c_2 y_2$$