Differential Equation

Lecture 2

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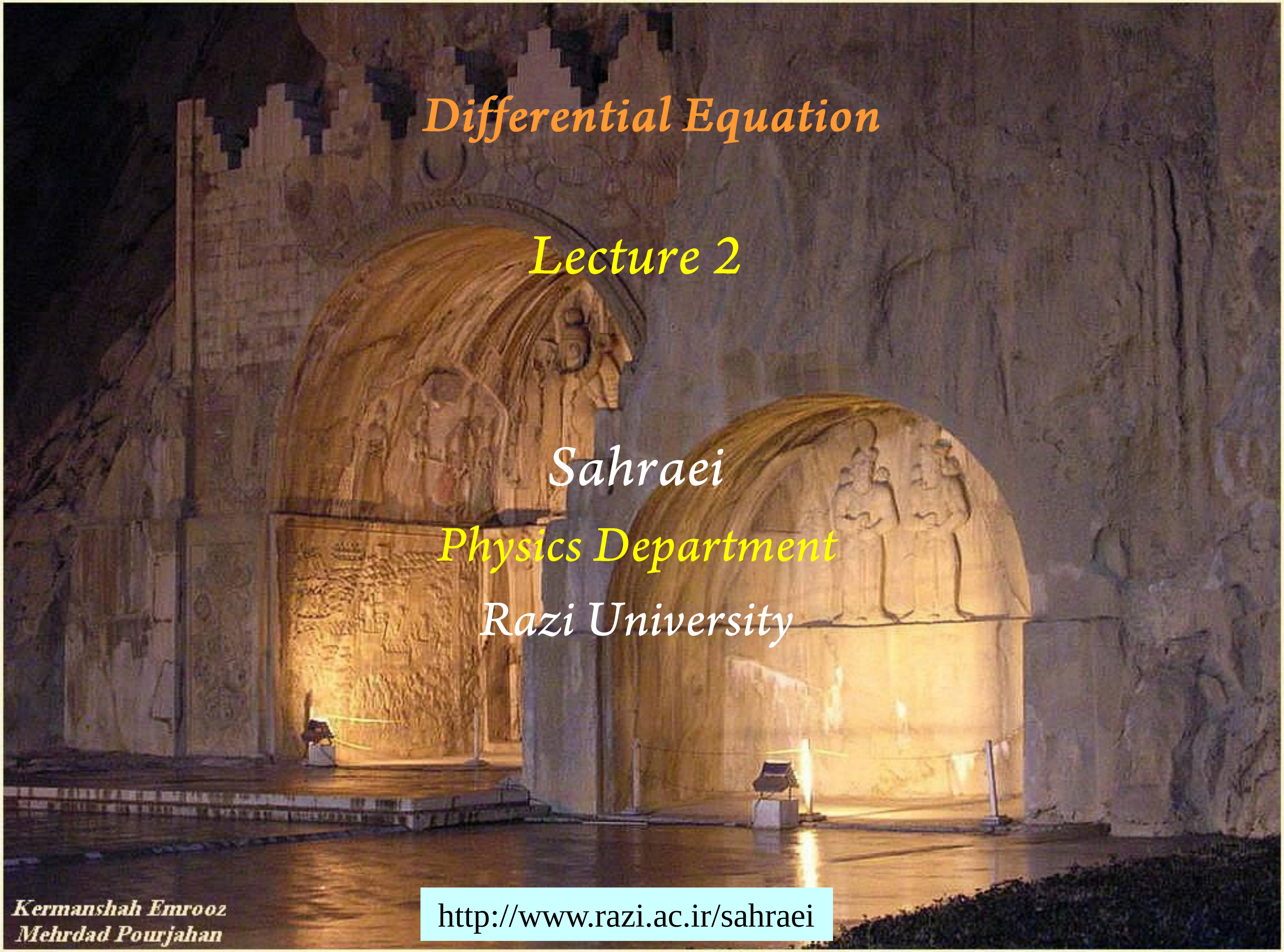

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Review of Lesson 1

- **Differential Equations Definition** $$\frac{dy}{dt} = ky(t)$$
- **Applications of differential equations**
- **Types of differential equations** $$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$$
- **Odinary Differential Equations**
- **Partial Differential Equations** $$m\frac{d^2 x}{dt^2} = -kx$$
- **Order of a Differential Equation**
- **Degree of a Differential Equation** $$y'' + xy^2 (dy/dx)^3 = e^x$$
- **Linear Differential Equation** $$\frac{dy}{dx} = a(x)y + b(x)$$
- **Nonlinear Differential Equation**
- **Homogeneous Nonhomogeneous Differential Equation** $$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P$$

General Remarks on Solutions

Does a differential equation have a solution?

$$F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}) = 0$$

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0 \quad \rightarrow y = y(x, c_1, c_2, \ldots, c_n)$$

$$F(x, y, y', y'') = 0 \quad \rightarrow y = y(x, c_1, c_2)$$

$$F(x, y, y') = 0 \quad \rightarrow y = y(x, c)$$

General Solution: **Solutions obtained from integrating the differential equations are called general solutions. The general solution of a order ordinary differential equation contains arbitrary constants resulting from integrating times.**

$$\frac{dv}{dt} = 2t + 4 \qquad dv = (2t+4)dt$$

$$\int dv = \int (2t+4)dt \qquad v + c_1 = t^2 + 4t + c_2$$

$$v_g = t^2 + 4t + c$$ **General Solution**

C is arbitrary constants

Particular Solution: **Particular solutions are the solutions obtained by assigning specific values to the arbitrary constants in the general solutions.**

$$v_g = t^2 + 4t + c$$

Initial Condition $t = 0, v = 0 \rightarrow c = 0$

$$t = 0, v = 2m/s \rightarrow c = 2m/s$$

$$v_p = t^2 + 4t + 2$$

Initial Condition: **Constrains that are specified at the initial point, generally time point, are called initial conditions. Problems with specified initial conditions are called initial value problems.**

Conditions

- **Boundary Condition**: Constrains that are specified at the boundary points, generally space points, are called boundary conditions. Problems with specified boundary conditions are called boundary value problems.

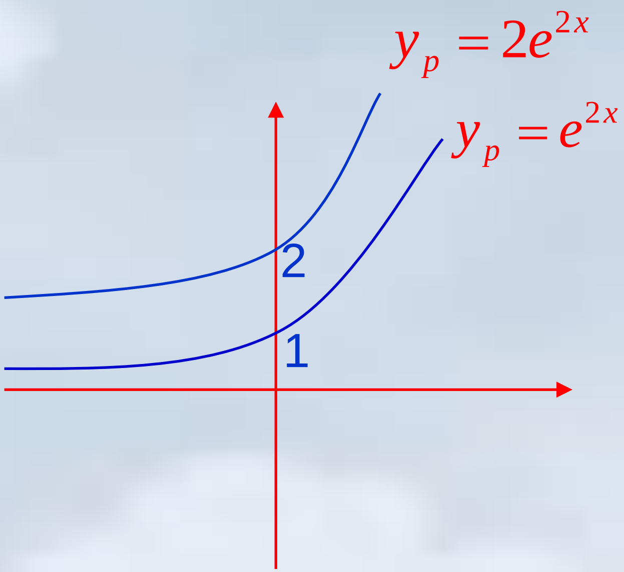
Example

$$y'' - 5y' + 6y = 0$$

$$y_g = c_1 e^{2x} + c_2 e^{3x} \qquad y_p = e^{2x}$$

$$y' = 2e^{2x} \quad , \qquad y'' = 4e^{2x}$$

$$4e^{2x} - 10e^{2x} + 6e^{2x} = 0$$



Integral Curves and Differential Equations

$$F(x, y, \frac{dy}{dx}) = 0 \qquad \frac{dy}{dx} = f(x, y)$$

The geometric meaning of a solution

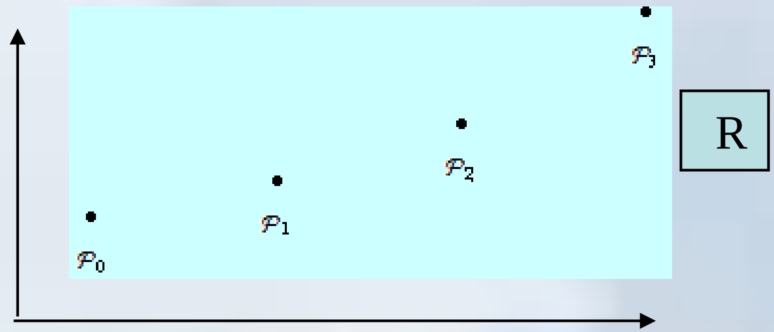
If this function f(x,y) is continuous throughout some region R in the x-y plane, we can represent a solution of the form

$$(\frac{dy}{dx})_{p_0} = f(x_0, y_0)$$

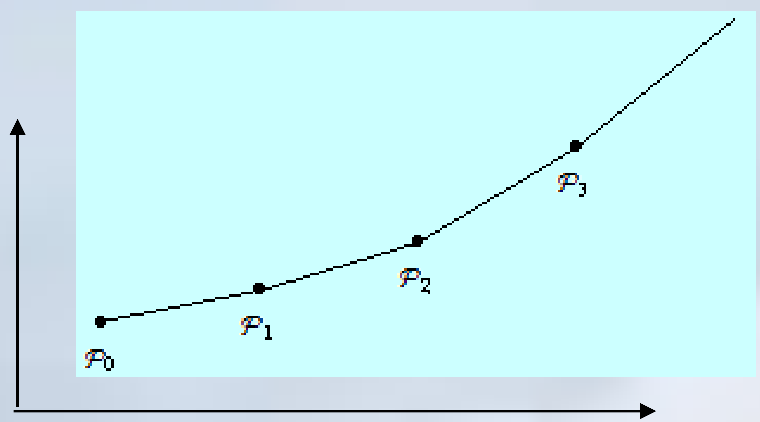
for a point p_0 within the region R . This solution determines a direction (the tangent of the solution at the point). We can choose another point p_1 within the region near p_0 such that

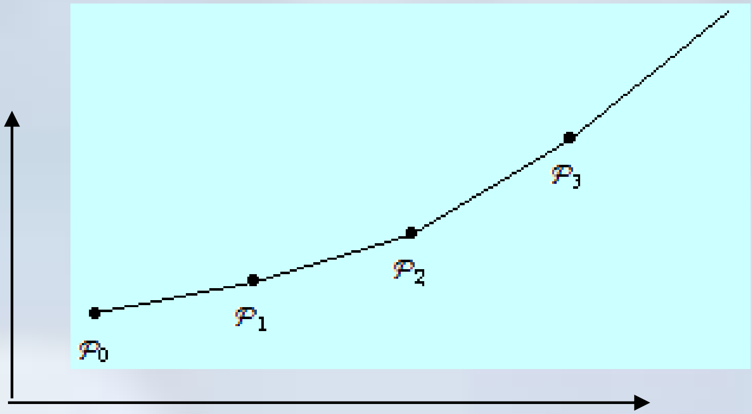
$$\left(\frac{dy}{dx}\right)_{p_1} = f(x_1, y_1)$$

We can continue this process until we get something that looks like this.



We could link these points by line segments so that we get a broken curve.





If we bring the points closer and closer together we will eventually get a smooth curve. If we think about this long enough we will see that if we start at a different initial point, we will get a different curve. In this way the solution of a differential equation in general will produce a family of curves dependent upon the initial point. Such a curve is called an *integral curve* since the process of solving a differential equation usually involves integration. The initial point is, in part, determined by the value of the constant c discussed earlier. Such a constant is called a *parameter*.

Picard's Theorem: If *f(x,y)* and $\partial f / \partial y$ are continuous functions on a closed regio R , then through each point (x_0,y_0) in the interior of the region there will pass a unique integral curve of the differential equation *dy/dx=f(x,y)*.

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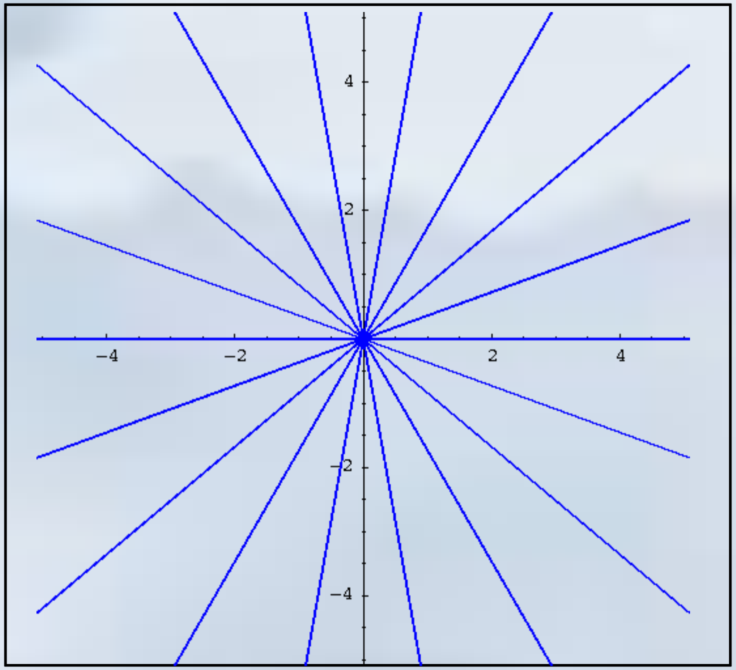
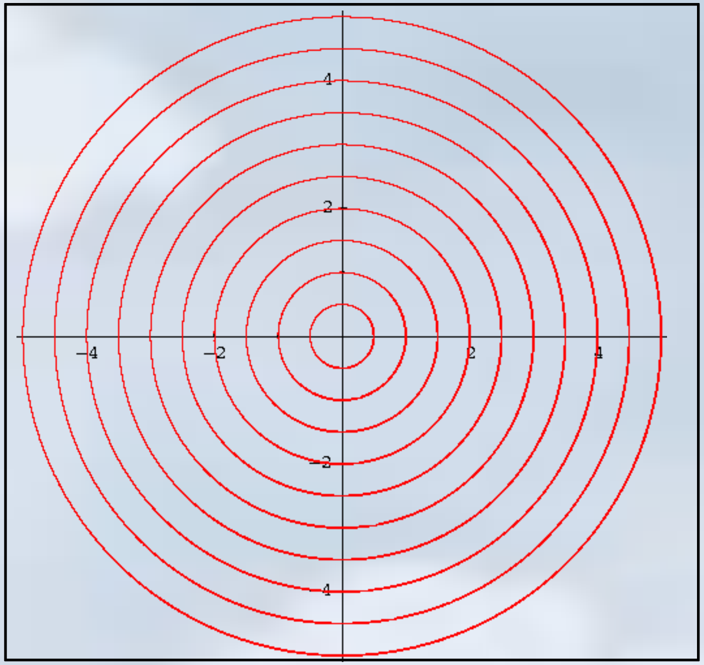
Families of Curves. Orthogonal Trajectories

$$y = mx \qquad\qquad y^2 + x^2 = c^2$$

Note that the first family describes all the lines passing by the origin (0,0) while the second the family describes all the circles centered at the origin (including the limit case when the radius 0 which reduces to the single point (0,0)) (see the pictures below).





In this page, we will only use the variables *x* and *y*. Any family of curves will be written as

$$f(x, y, c) = 0$$

One may ask whether any family of curves may be generated from a differential equation? In general, the answer is no. Let us see how to proceed if the answer were to be yes. First differentiate with respect to *x*, and get a new equation involving in general *x*, *y*, *dy/dx* , and *C*. Using the original equation, we may able to eliminate the parameter *C* from the new equation.

روش بدست آوردن مسیرهای قائم یک دسته منحنی

$$y^2 + x^2 = c^2$$

1- تشکیل معادله دیفرانسیل مسیر اصلی

$$2y\frac{dy}{dx} + 2x = 0 \qquad \frac{dy}{dx} = -\frac{x}{y}$$

2- بجای dy/dx ، $-\frac{1}{dy/dx}$ قرار می دهیم

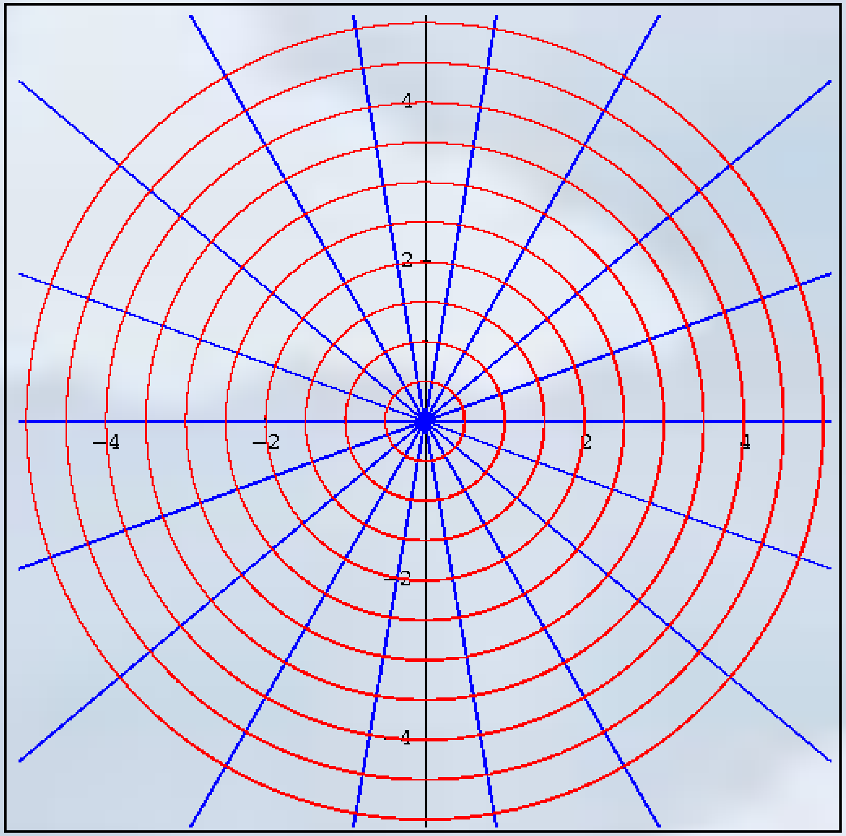
$$-\frac{1}{dy/dx} = -\frac{x}{y}$$

3- معادله دیفرانسیل مسیر قائم را حل می کنیم

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\ln y = \ln x + \ln m$$

$$y = mx$$



Example: Find the orthogonal family to the family of circles

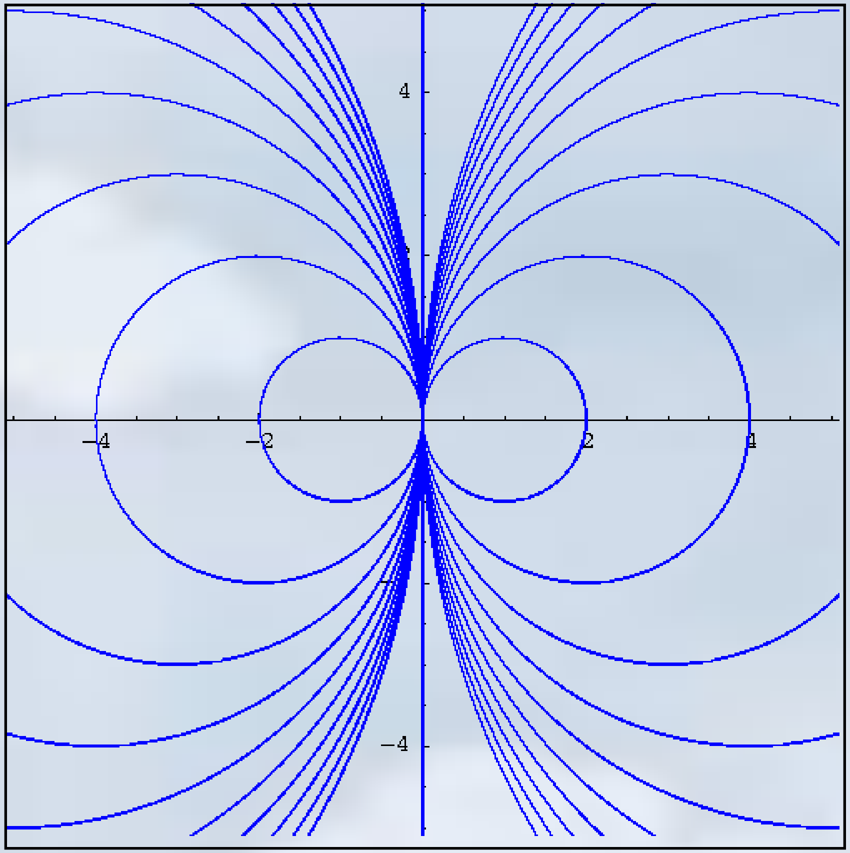
$$x^2 + y^2 = 2cx$$

$$2x + 2y\frac{dy}{dx} = 2c$$

$$2x + 2y\frac{dy}{dx} = \frac{x^2 + y^2}{x}$$

$$x^2 + 2xy\frac{dy}{dx} = y^2$$

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$



$$\frac{dy}{dx}=\frac{y^2-x^2}{2xy}$$

$$\frac{dy}{dx}\rightarrow-\frac{1}{dy/dx}$$

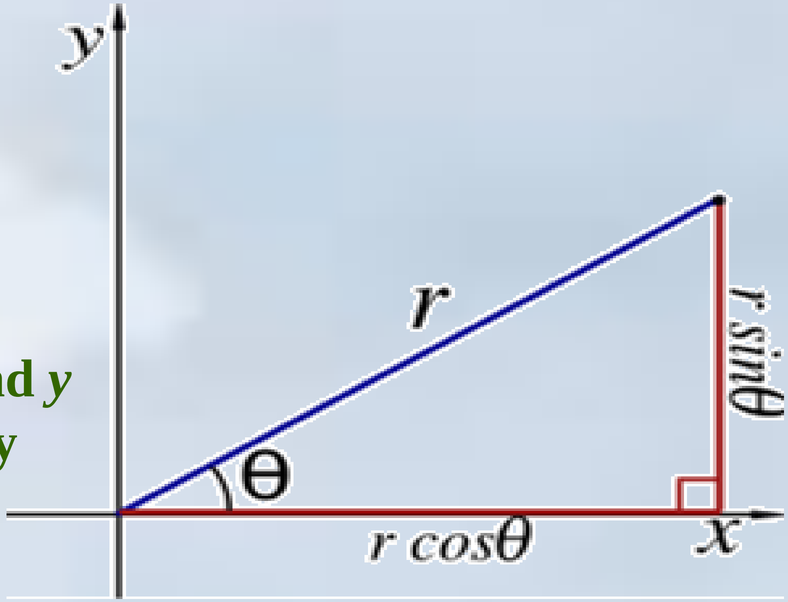
$$\frac{dy}{dx}=\frac{2xy}{x^2-y^2}$$

Converting between polar and Cartesian coordinates

The two polar coordinates *r* and θ can be converted to the Cartesian coordinates *x* and *y* by using the trigonometric functions *sine* and *cosine*:

$$x = r\cos\theta$$

$$y = r\sin\theta$$



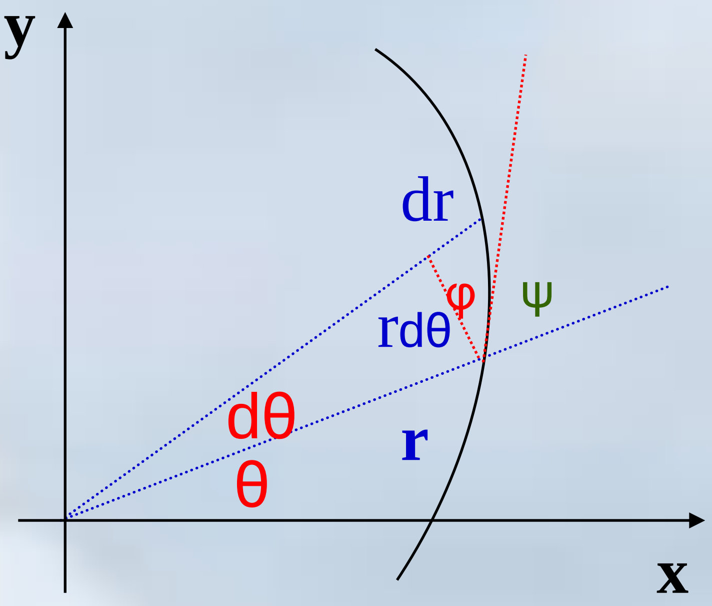
while the two Cartesian coordinates *x* and *y* can be converted to polar coordinate *r* by

$$r = \sqrt{x^2 + y^2}$$

$$\theta = tg^{-1}\left(\frac{y}{x}\right)$$

$$\psi + \varphi = \frac{\pi}{2}$$

$$tg\psi = \cot\varphi = \frac{rd\theta}{dr}$$



The angle between the tangent and radial line at the point (r, θ) is

$$tg\psi = \frac{dy}{dx} = \frac{rd\theta}{dr}$$

$$\frac{dy}{dx} = \frac{-1}{rd\theta / dr}$$

$$x^2 + y^2 = 2cx$$

$$r^2\cos^2\theta + r^2\sin\theta = 2cr\cos\theta$$

$$r = 2c\cos\theta$$

$$\frac{dr}{d\theta} = -2c\sin\theta$$

$$\frac{dr}{d\theta} = -\frac{r}{\cos\theta}\sin\theta$$

$$\frac{rd\theta}{dr} = -\frac{\cos\theta}{\sin\theta}$$

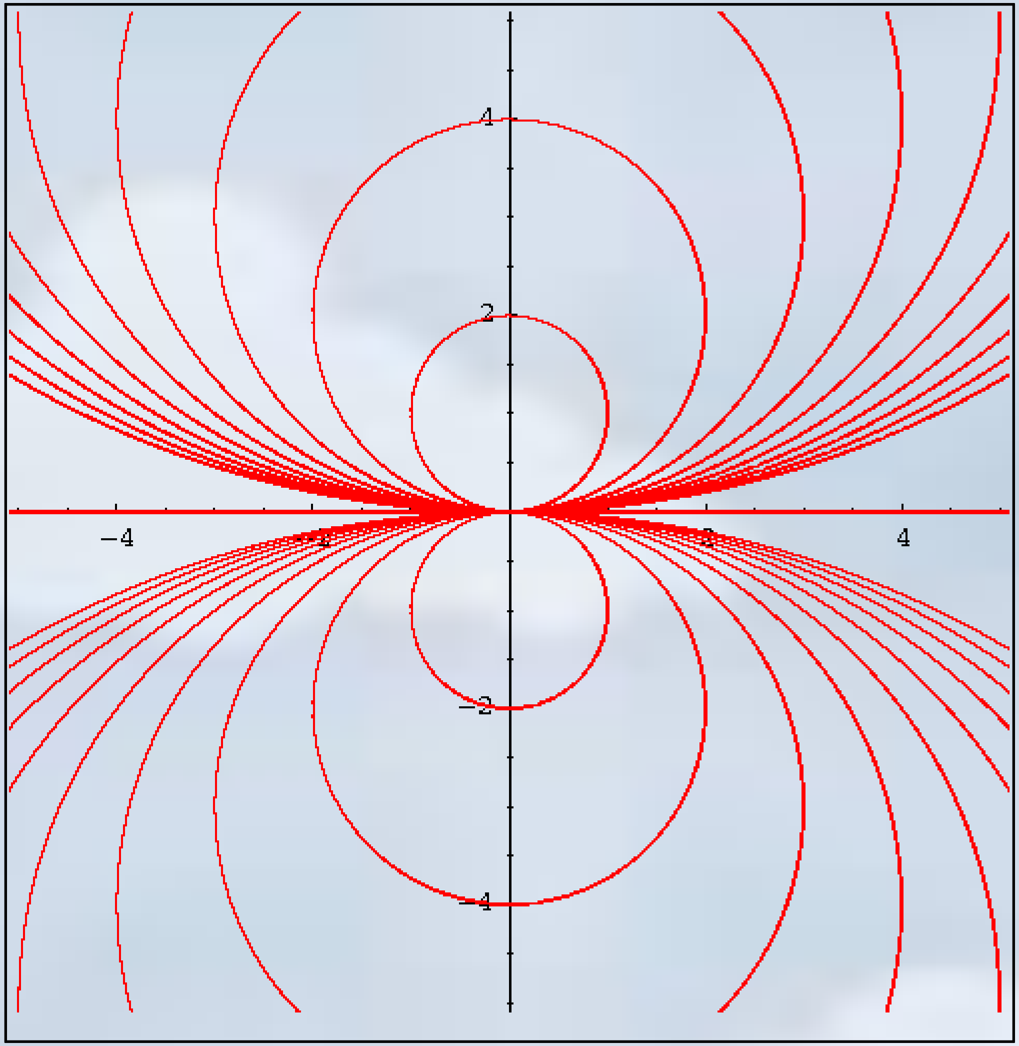
$$\frac{rd\theta}{dr} = \frac{\sin\theta}{\cos\theta}$$

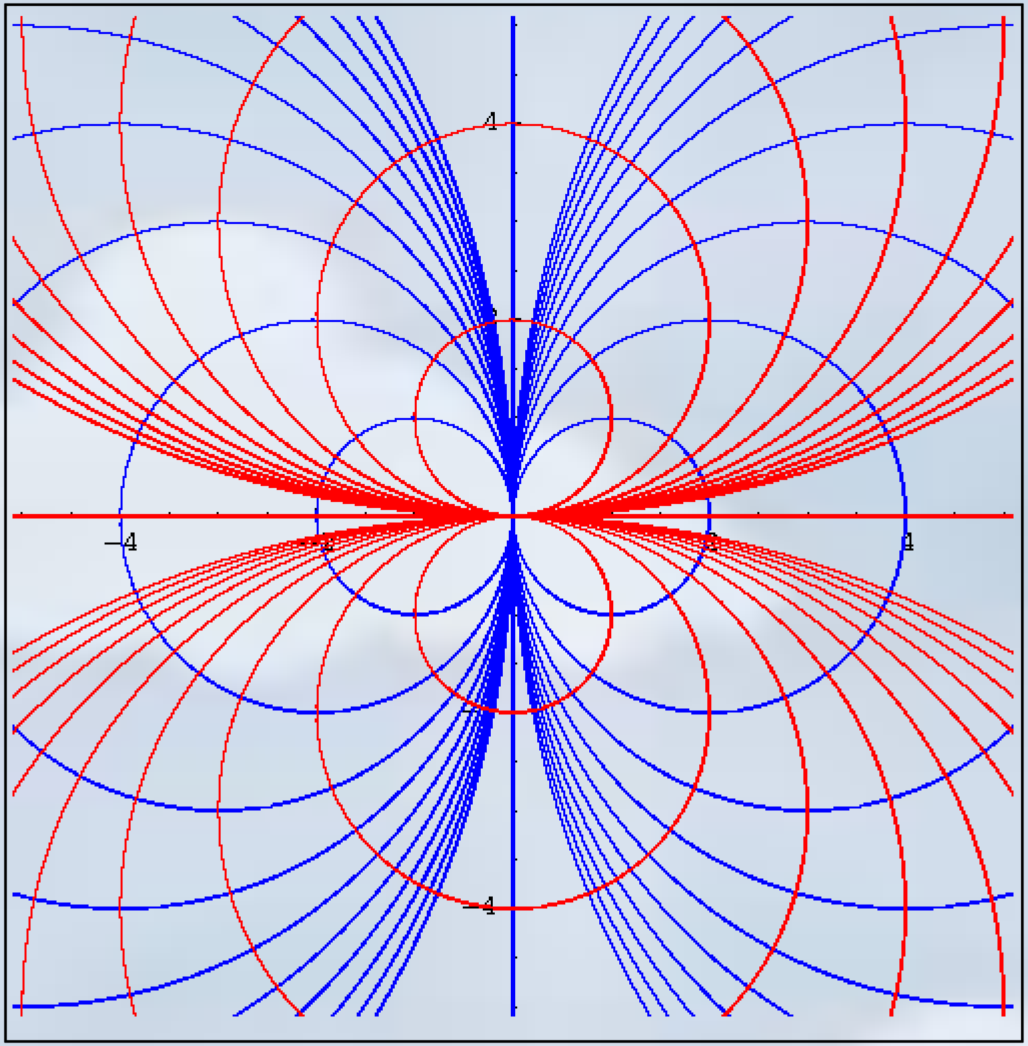
$$\frac{dr}{r} = \frac{\cos\theta d\theta}{\sin\theta}$$

$$\ln r = \ln\sin + \ln c$$

$$r = c\sin\theta$$

$$r = c \sin \theta$$





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Linear Equation Applications

Growth, Decay and Chemical Reaction

$$-\frac{dx}{dt} = kx \quad , \quad \mathrm{k} > 0 \qquad \frac{dx}{x} = -kdt$$

$$\ln x = -kt + c$$

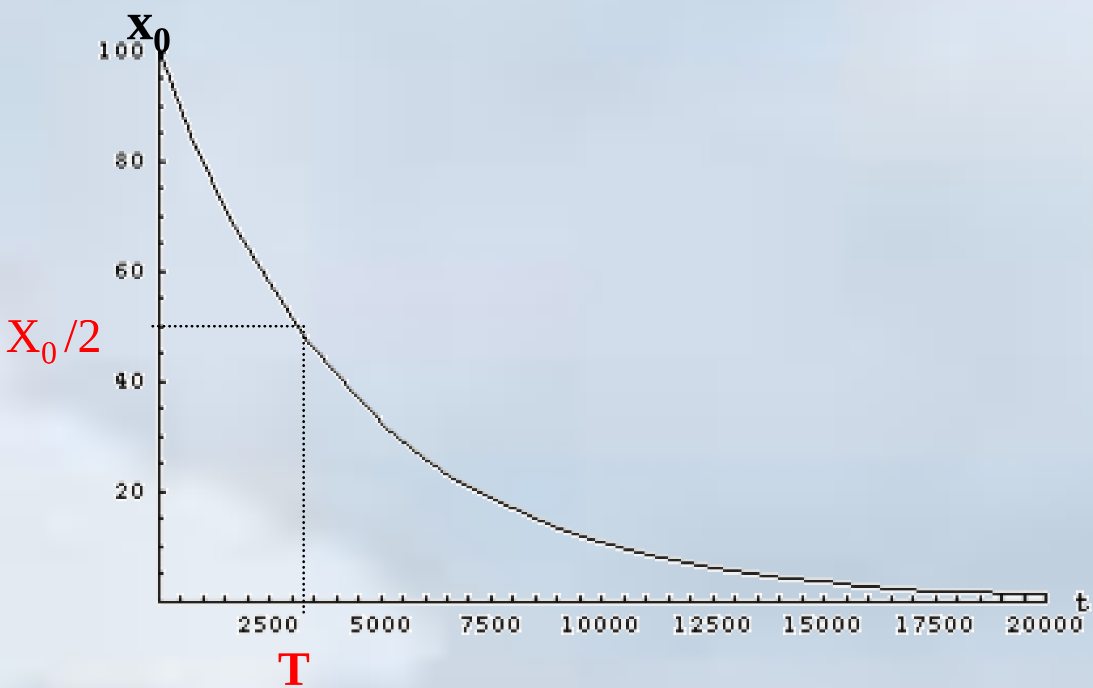
$$t = o \ \text{ when } \ x = x_0 \quad \rightarrow c = \ln x_0$$

$$\ln x = -kt + \ln x_0$$

$$\ln \frac{x}{x_0} = -kt \qquad x = x_0 e^{-kt}$$

$$\frac{x_0}{2} = x_0 e^{-kt}$$

$$kT = \ln 2$$



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Thanks For Your Attention