Differential Equations

Lecture 19

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Legendre Equation

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0$$ ***P is a constant***

since $p(x) = -2x/(1 - x^2)$ and

$q(x) = p\,(p+1)/(1 - x^2)$ are analytic at $x_0 = 0$.

Thus $x_0 = 0$ is an ordinary point,

Also, *p* and *q* have singular points at $x_0 = \pm 1$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x^2 \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} +$$

$$p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n} - 2 \sum_{n=1}^{\infty} n a_n x^{n} +$$

$$p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^{n} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n} - 2 \sum_{n=1}^{\infty} n a_n x^{n} +$$

$$p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + p(p+1)a_n = 0$$

$$(n+2)(n+1)a_{n+2} + \left[-n(n-1) - 2n + p(p+1)\right]a_n = 0$$

$$(n+2)(n+1)a_{n+2} + \left[-n^2 + n - 2n + p^2 + p\right]a_n = 0$$

$$a_{n+2} = -\frac{(p-n)(p+n+1)}{(n+2)(n+1)}a_n$$

Just as in the previous example, this recursion formula enable us to express a_n in terms of a_0 or a_1 according as n is even or odd:

$$n = 0 \rightarrow a_2 = -\frac{p(p+1)}{2\times 1}a_0$$

$$a_{n+2} = -\frac{(p-n)(p+n+1)}{(n+2)(n+1)} a_0$$

$$n=1 \rightarrow a_3 = -\frac{(p-1)(p+2)}{3\times 2} a_1$$

$$n=2 \rightarrow a_4 = -\frac{(p-2)(p+3)}{4\times 3} a_2 \quad = \frac{p(p-2)(p+1)(p+3)}{4!} a_0$$

$$n=3 \rightarrow a_5 = -\frac{(p-3)(p+4)}{5\times 4} a_3$$

$$= \frac{(p-1)(p-3)(p+2)(p+4)}{5!} a_1$$

$$n = 4 \rightarrow a_6 = -\frac{(p-4)(p+5)}{6 \times 5} a_4 \qquad a_{n+2} = -\frac{(p-n)(p+n+1)}{(n+2)(n+1)} a_0$$

$$= -\frac{p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!} a_0$$

$$n = 5 \rightarrow a_7 = -\frac{(p-5)(p+6)}{7 \times 6} a_5$$

$$= -\frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{7!} a_1$$

And so on. By inserting these coefficients into the assumed solution, we obtain.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

$$y = a_0\left[1 - \frac{p(p+1)}{2!}x^2 + \frac{p(p-2)(p+1)(p+3)}{4!}x^4\right.$$

$$\left. - \frac{p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!}x^6 + \ldots\right]$$

$$+ a_1\left[x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p-3)(p+2)(p+4)}{5!}x^5\right.$$

$$\left. - \frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{7!}x^7 + \ldots\right]*$$

The function defined by * are called *Legendre functions*.

$$y_g(x) = a_0 y_1(x) + a_1 y_2(x)$$

$$a_{n+2} = -\frac{(p-n)(p+n+1)}{(n+2)(n+1)} a_n$$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+2}} \right| = \lim_{n\to\infty} \left| \frac{(n+2)(n+1)}{(p-n)(p+n+1)} \right| = 1$$

a) When *p* is not an integer, both the two solutions have infinite number of terms.

(b) When *p* is an even integer, $y_1(x)$ has finite number of terms and $y_2(x)$is a series.

(c) When *p* is an odd integer, $y_2(x)$ has finite number of terms and $y_1(x)$ is a series.

$y_1(x)$ when *p* is an even integer and $y_2(x)$ when *n* is an odd integer are called the Legendre polynomials (denoted by $P_n(x)$).

The Legendre Polynomials *$p_n(x)$* can be expressed by *Rodrigues' formula*. It provides a relatively easy method for computing the successive Legendre polynomials, of which first few are:

$$p_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \qquad n = 0,1,2,3...$$

$$p_0(x) = 1$$

$$p_1(x) = x$$

$$p_2(x) = \frac{1}{2}(3x^2-1)$$

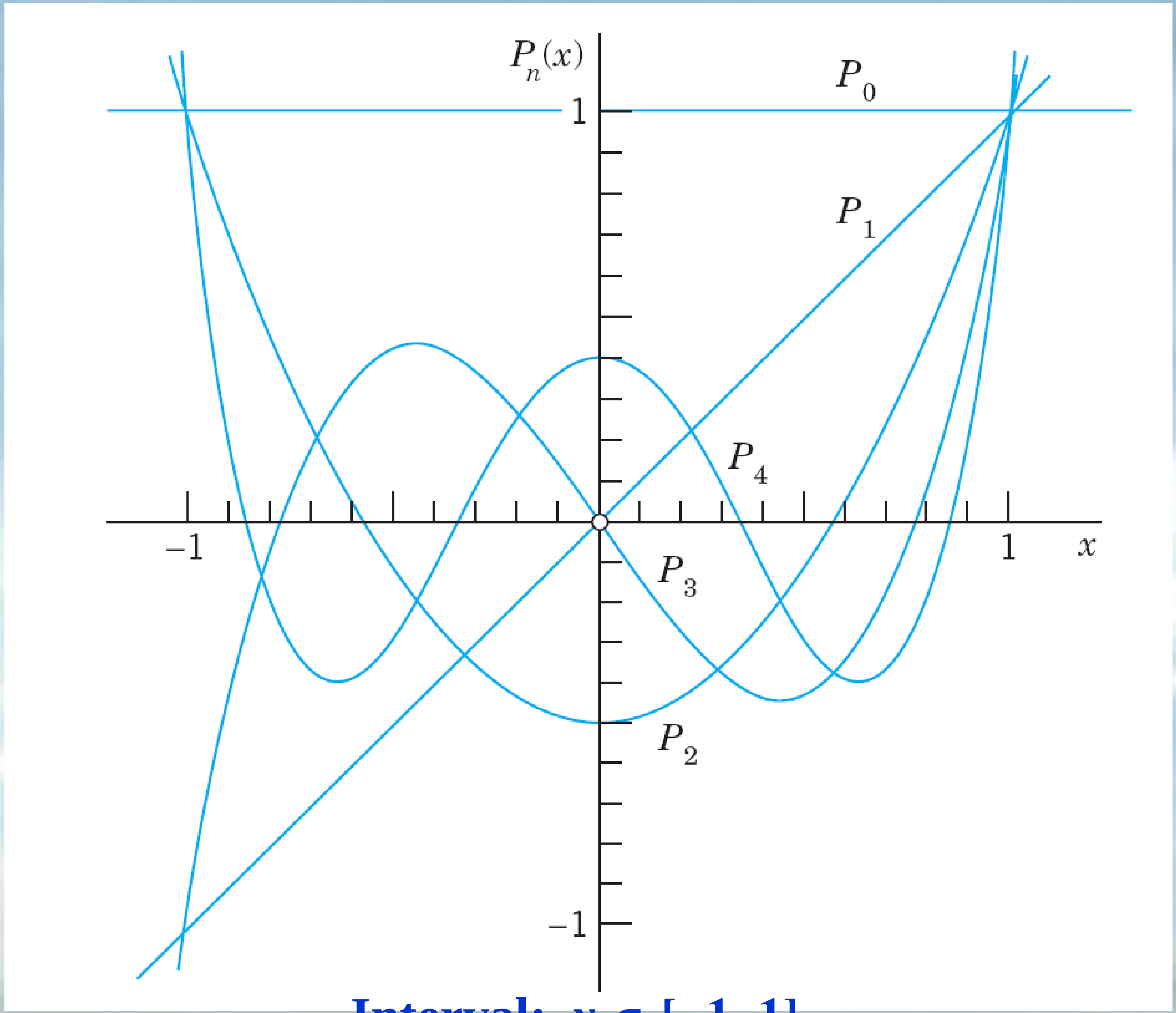
$$p_3(x) = \frac{1}{2}(5x^3-3x)$$

$$p_4(x) = \frac{1}{8}(35x^4-30x^2+3)$$

If *n* is even we only have even powers of x, and only odd powers if *n* is odd

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2 - 1)$
3	$\frac{1}{2}(5x^3 - 3x)$
4	$\frac{1}{8}(35x^4 - 30x^2 + 3)$
5	$\frac{1}{8}(63x^5 - 70x^3 + 15x)$
6	$\frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$
7	$\frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$
8	$\frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)$
9	$\frac{1}{128}(12155x^9 - 25740x^7 + 18018x^5 - 4620x^3 + 315x)$
10	$\frac{1}{256}(46189x^{10} - 109395x^8 + 90090x^6 - 30030x^4 + 3465x^2 - 63)$

Legendre polynomials



Interval: $x \in [-1, 1]$

Some useful properties:

Orthogonality: The most important property of the Legendre polynomials is the fact that $-1 \le x \le 1$

$$\int_{-1}^{+1} P_m(x) P_n(x)\,dx = \begin{cases} 0 & \text{if } m \ne n \\ \dfrac{2}{2n+1} & \text{if } m = n \end{cases}$$

$$1 = p_0(x), \qquad x = p_1(x),$$

$$p_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$x^2 = \frac{1}{3} + \frac{2}{3} p_2(x) = \frac{1}{3} p_0(x) + \frac{2}{3} p_2(x),$$

$$p_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$x^3 = \frac{3}{5} x + \frac{2}{5} p_3(x) = \frac{3}{5} p_1(x) + \frac{2}{5} p_3(x)$$

$$p(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$$

$$p(x) = b_0 p_0(x) + b_1 p_1(x) + b_2 \left[\frac{1}{3} p_0(x) + \frac{2}{3} p_2(x) \right]$$

$$+ b_3 \left[\frac{3}{5} p_1(x) + \frac{2}{5} p_3(x) \right]$$

$$p(x) = b_0 p_0(x) + b_1 p_1(x) + b_2\left[\frac{1}{3} p_0(x) + \frac{2}{3} p_2(x)\right]$$

$$+ b_3\left[\frac{3}{5} p_1(x) + \frac{2}{5} p_3(x)\right]$$

$$= (b_0 + \frac{b_2}{3}) p_0(x) + (b_1 + \frac{3b_3}{5}) p_1(x) + \frac{2b_2}{3} p_2(x) + \frac{2b_3}{5} p_3(x)$$

$$= \sum_{n=0}^{3} a_n p_n(x)$$

$$p(x) = \sum_{n=0}^{k} a_n p_n(x)$$

$$f(x)=\sum_{n=0}^{\infty}a_n p_n(x)$$

$$\int_{-1}^{1} f(x)\,p_m(x)dx=\sum_{n=0}^{\infty}a_n\int_{-1}^{1} p_m(x)\,p_n(x)dx$$

$$=a_n\int_{-1}^{1} p_m(x)\,p_n(x)dx$$

$$a_n=\frac{\int_{-1}^{1} f(x)\,p_m(x)dx}{\int_{-1}^{1} p_m(x)\,p_n(x)dx}=\frac{\int_{-1}^{1} f(x)\,p_n(x)dx}{\dfrac{2}{2n+1}}$$

$$a_n=(n+\frac{1}{2})\int_{-1}^{1} f(x)\,p_n(x)dx$$

Show the following polynomials in terms of the Legendre polynomials

$$f(x) = 5x^3 - 3x^2 - x - 1$$

$$5x^3 - 3x^2 - x - 1 = a_0 p_0(x) + a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x)$$

$$a_n = (n + \frac{1}{2})\int_{-1}^{1} f(x) p_n(x) dx$$

$$a_0 = \frac{1}{2}\int_{-1}^{1}(5x^3 - 3x^2 - x - 1)(1)dx = \frac{1}{2}(-2 - 2) = -2$$

$$a_1 = \frac{3}{2}\int_{-1}^{1}(5x^3 - 3x^2 - x - 1)(x)dx = \frac{3}{2}(2 - \frac{2}{3}) = 2$$

$$a_2 = \frac{5}{2}\int_{-1}^{1}(5x^3 - 3x^2 - x - 1)(\frac{3}{2}x^2 - \frac{1}{2})dx$$

$$= \frac{5}{2}(-\frac{9}{5} + 1) = -2$$

$$a_3 = \frac{2}{7}\int_{-1}^{1}(5x^3 - 3x^2 - x - 1)(\frac{5}{2}x^3 - \frac{3}{2}x)dx = 2$$

$$5x^3 - 3x^2 - x - 1 = 2(-p_0(x) + p_1(x) - p_2(x) + p_3(x))$$

Singular points

$$y'' + P(x)y' + Q(x)y = 0 \; *$$

P(x) **and** ***Q(x)*** **are analytic at the x_0 and therefore have power series expansions. In these cases x_0 is called ordinary point of equation.**

$$P(x) = \sum_{n=0}^{\infty} p_n x^n = p_0 + p_1 x + p_2 x^2 + \ldots$$

$$Q(x) = \sum_{n=0}^{\infty} q_n x^n = q_0 + q_1 x + q_2 x^2 + \ldots$$

Equation also is analytic

$$Example \quad y'' + 2xy' - \frac{1}{x} y = 0 \qquad P(x) = 2x, \quad Q(x) = -\frac{1}{x}$$

$$x_0 = 1 \rightarrow ordinary \quad point$$

$$x_0 = 0 \rightarrow singular \quad point$$

Singular points are regular or irregular

We recall that a point x_0 is a singular point of the differential equation

$$y'' + P(x)y' + Q(x)y = 0 \quad *$$

if one or the other (or both) of the coefficient functions P(x) and Q(x) fails to be analytic at x_0.

A singular point x_0 of equation * is said to be regular if the function $(x-x_0)P(x)$ *and* $(x-x_0)^2Q(x)$ are analytic, and irregular otherwise.

If x_0 is an ordinary point, then *p* and *q* are analytic and have derivatives of all orders at x_0, and this enables us to solve for a_n in the solution expansion $y(x) = \Sigma\, a_n(x - x_0)^n$.

Example: Consider the Legendre equation

$$y'' - \frac{2x}{\left(1-x^2\right)} y' + \frac{p\left(p+1\right)}{\left(1-x^2\right)} y = 0$$

It is clear that *x=1* and *x=-1* are singular points. The first is regular because

$$(x-1)\,p(x) = \frac{2x}{x+1} \qquad (x-1)^2 Q(x) = -\frac{(x-1)\,p(p-1)}{x+1}$$

are analytic at *x=1*, and second is also regular.

$$(x+1)\,p(x) = \frac{2x}{1-x} \qquad (x-1)^2 Q(x) = -\frac{(1+x)\,p(p-1)}{1-x}$$

Differential Equation Solution about regular singular point

If x=0 is a regular singular point and differential equations is solved by the Frobenius method.

$$y'' + P(x)y' + Q(x)y = 0$$

Frobenius Method to solve the second-order ODEs having coefficients being not analytic.

$$y = x^m \sum_{n=0}^{\infty} a_n x^n = x^m (a_0 + a_1 x + a_2 x^2 + \ldots)$$

$$= a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \ldots$$

where $x_0=0$ is a regular singular point and the exponent m may be a negative integer, a fraction, or even an irrational real number.

Euler Equations

A relatively simple differential equation that has a regular singular point is the Euler equation,

$$x^2 y'' + p\,xy' + q\,y = 0$$ **where *p*, *q* are constants.**

Note that $x_0 = 0$ is a regular singular point.

The solution of the Euler equation is typical of the solutions of all differential equations with regular singular points,

Series Solutions Near a Regular Singular Point

We now consider solving the general second order linear equation in the neighborhood of a regular singular point x_0. For convenience, will will take $x_0 = 0$.

Recall that the point $x_0 = 0$ is a regular singular point of

$$y'' + P(x)y' + Q(x)y = 0$$

$$xP(x) \text{ and } x^2Q(x) \text{ are analytic at } x = 0$$

$$p(x) = xP(x) = \sum_{n=0}^{\infty} p_n x^n$$

$$q(x) = x^2Q(x) = \sum_{n=0}^{\infty} q_n x^n$$

$$y'' + P(x)y' + Q(x)y = 0$$

multiplying by x^2, we obtain

$$x^2y'' + x^2P(x)y' + x^2Q(x)y = 0$$

$$x^2y'' + x\left[xP(x)\right]y' + \left[x^2Q(x)\right]y = 0$$

$$x^2y'' + xp(x)y' + q(x)\,y = 0$$

$$x^2y'' + x\left(p_0 + p_1x + p_2x^2 + \cdots\right)y' + \left(q_0 + q_1x + q_2x^2 + \cdots\right)y = 0$$

$$y'' + \left(\frac{p_0 + p_1x + p_2x^2 + \cdots}{x}\right)y' + \left(\frac{q_0 + q_1x + q_2x^2 + \cdots}{x^2}\right)y = 0$$

Note that if $p_1 = p_2 = \cdots = q_1 = q_2 = \cdots = 0$

then our differential equation reduces to the Euler Equation

$$x^2 y'' + p_0 x y' + q_0 y = 0$$

In any case, our equation is similar to an Euler Equation but with power series coefficients.

Thus our solution method: assume solutions have the form

$$y(x) = x^m \left(a_0 + a_1 x + a_2 x^2 + \cdots \right) = \sum_{n=0}^{\infty} a_n x^{m+n}$$

$$= a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \ldots$$

Example : Regular Singular Point

$$2x^2 y'' - xy' + (1+x)y = 0$$

$$x^2 y'' - \frac{x}{2} y' + \frac{1+x}{2} y = 0$$

Since the coefficients are polynomials, it follows that *x* = 0 is a regular singular point, since both limits below are finite:

$$\lim_{x \to 0} x\left(-\frac{x}{2x^2}\right) = -\frac{1}{2} < \infty \qquad \lim_{x \to 0} x^2\left(\frac{1+x}{2x^2}\right) = \frac{1}{2} < \infty$$

Now $xp(x) = -1/2$ and $x^2q(x) = (1 + x)/2$, and thus for

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \qquad x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n,$$

$$p_0 = -1/2,\ q_0 = 1/2,\ q_1 = 1/2, \quad p_1 = p_2 = \cdots = q_2 = q_3 = \cdots = 0$$

Thus the corresponding Euler Equation is

$$x^2 y'' + p_0 x y' + q_0 y = 0$$

$$2x^2 y'' - x y' + y = 0$$

$$y(x) = x^m\left(a_0 + a_1 x + a_2 x^2 + \cdots\right) = \sum_{n=0}^{\infty} a_n x^{m+n}$$

$$= a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \ldots$$

$$y'(x) = \sum_{n=0}^{\infty} a_n (m+n) x^{m+n-1}$$

$$= a_0 m x^{m-1} + a_1 (m+1) x^m + a_2 (m+2) x^{m+1} + \ldots$$

$$y''(x) = \sum_{n=0}^{\infty} a_n (m+n)(m+n-1) x^{m+n-2}$$

$$= a_0 m(m-1)x^{m-2} + a_1(m+1)mx^{m-1} + a_2(m+2)(m+1)x^m + \ldots$$

By substitution, our differential equation becomes

$$2x^2 y'' - xy' + (1+x)y = 0$$

$$\sum_{n=0}^{\infty} 2a_n (m+n)(m+n-1) x^{m+n} - \sum_{n=0}^{\infty} a_n (m+n) x^{m+n}$$

$$+ \sum_{n=0}^{\infty} a_n x^{m+n} + \sum_{n=0}^{\infty} a_n x^{m+n+1} = 0$$

$$\sum_{n=0}^{\infty} 2a_n(m+n)(m+n-1)x^{m+n} - \sum_{n=0}^{\infty} a_n(m+n)x^{m+n}$$

$$+\sum_{n=0}^{\infty} a_n x^{m+n} + \sum_{n=1}^{\infty} a_{n-1} x^{m+n} = 0$$

$$a_0[2m(m-1)-m+1]x^m$$

$$+\sum_{n=1}^{\infty}\{a_n[2(m+n)(m+n-1)-(m+n)+1]+a_{n-1}\}x^{m+n} = 0$$

$$a_0[2m(m-1)-m+1]=0$$

$$a_n[2(m+n)(m+n-1)-(m+n)+1]+a_{n-1}=0, \quad n=1,2,\ldots$$

$$a_0\left[2m(m-1)-m+1\right]=0, \quad a_0 \neq 0$$

The equation is called the **indicial equation**, and was obtained earlier when we examined the corresponding Euler Equation.

$$2m^2-3m+1=(2m-1)(m-1)=0$$

The roots m_1 = 1, m_2 =1/2, of the indicial equation are called the exponents of the singularity, for regular singular point x = 0.

The exponents of the singularity determine the qualitative behavior of solution in neighborhood of regular singular point.

$$a_n\left[2(m+n)(m+n-1)-(m+n)+1\right]+a_{n-1}=0$$

$$a_n = -\frac{a_{n-1}}{2(m+n)(m+n-1) - (m+n) + 1}$$

$$= -\frac{a_{n-1}}{2(m+n)^2 - 3(m+n) + 1}$$

$$a_n = -\frac{a_{n-1}}{[2(m+n)-1][(m+n)-1]}, \quad n \geq 1$$ **m_1 = 1, m_2 =1/2,**

Starting with m_1 = 1, this recursion becomes

$$a_n = -\frac{a_{n-1}}{[2(1+n)-1][(1+n)-1]} = -\frac{a_{n-1}}{(2n+1)n}, \quad n \geq 1$$

$$a_n = -\frac{a_{n-1}}{(2n+1)n} \qquad a_1 = -\frac{a_0}{3\times 1}$$

$$a_2 = -\frac{a_1}{5\times 2} = \frac{a_0}{(3\times 5)(1\times 2)}$$

$$a_3 = -\frac{a_2}{7\times 3} = -\frac{a_0}{(3\times 5\times 7)(1\times 2\times 3)}, \ \cdots$$

$$a_n = \frac{(-1)^n a_0}{(3\times 5\times 7\cdots(2n+1))n!}, \ n \geq 1$$

Hence for *x* > 0, one solution to our differential equation is

$$y_1(x)=\sum_{n=0}^{\infty} a_n x^{n+m} = a_0 x + \sum_{n=1}^{\infty} \frac{(-1)^n a_0 x^{n+1}}{\big(3\times 5\times 7\cdots(2n+1)\big)n!}$$

$$= a_0 x\left[1+\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\big(3\times 5\times 7\cdots(2n+1)\big)n!}\right]$$

Thus if we omit a_0, one solution of our differential equation is

$$y_1(x)= x\left[1+\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\big(3\times 5\times 7\cdots(2n+1)\big)n!}\right], \quad x>0$$

To determine the radius of convergence, use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{\left(3\times5\times7\cdots(2n+1)\right)n!(-1)^{n+1}x^{n+1}}{\left(3\times5\times7\cdots(2n+1)(2n+3)\right)(n+1)!(-1)^n x^n}\right|$$

$$=\lim_{n\to\infty}\frac{|x|}{(2n+3)(n+1)}=0<1$$

Thus the radius of convergence is infinite, and hence the series converges for all *x*.

$$a_n=-\frac{a_{n-1}}{[2(m+n)-1][(m+n)-1]},\ \text{for } n\geq 1,\ m_1=1,\ \ m_1=1/2$$

When m_1 = 1/2, this recursion becomes

$$a_n = -\frac{a_{n-1}}{\left[2\left(1/2+n\right)-1\right]\left[\left(1/2+n\right)-1\right]}$$

$$= -\frac{a_{n-1}}{2n\left(n-1/2\right)} = -\frac{a_{n-1}}{n\left(2n-1\right)},\ n \geq 1$$

$$a_1 = -\frac{a_0}{1\times 1} \qquad a_2 = -\frac{a_1}{2\times 3} = \frac{a_0}{\left(1\times 2\right)\left(1\times 3\right)}$$

$$a_3 = -\frac{a_2}{3\times 5} = -\frac{a_0}{\left(1\times 2\times 3\right)\left(1\times 3\times 5\right)},\ \cdots$$

$$a_n = \frac{(-1)^n a_0}{\left(\left(1\times 3\times 5\right)\cdots\left(2n-1\right)\right)n!},\ n \geq 1$$

Hence for $x > 0$, a second solution to our equation is

$$y_2(x) = \sum_{n=0}^{\infty} a_n x^{n+1/2} = a_0 x^{1/2} + \sum_{n=1}^{\infty} \frac{(-1)^n a_0 x^{n+1/2}}{\left(1\times 3\times 5\cdots(2n-1)\right)n!}$$

$$= a_0 x^{1/2}\left[1+\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\left(1\times 3\times 5\cdots(2n-1)\right)n!}\right]$$

Thus if we omit a_0, the second solution is

$$y_2(x) = x^{1/2}\left[1+\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\left(1\times 3\times 5\cdots(2n-1)\right)n!}\right]$$

Radius of Convergence for Second Solution

To determine the radius of convergence for this series, we can use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\left(1\times 3\times 5\cdots(2n-1)\right)n!(-1)^{n+1}x^{n+1}}{\left(1\times 3\times 5\cdots(2n-1)(2n+1)\right)(n+1)!(-1)^n x^n}\right|$$

$$= \lim_{n\to\infty}\frac{|x|}{(2n+1)(n+1)} = 0 < 1$$

Thus the radius of convergence is infinite, and hence the series converges for all *x*.

General Solution

The two solutions to our differential equation are

$$y_1(x) = x\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{\left(3\times5\times7\cdots(2n+1)\right)n!}\right]$$

$$y_2(x) = x^{1/2}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{\left(1\times3\times5\cdots(2n-1)\right)n!}\right]$$

Since the leading terms of y_1 and y_2 are x and $x^{1/2}$, respectively, it follows that y_1 and y_2 are linearly independent, and hence form a fundamental set of solutions for differential equation.

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \quad x > 0,$$

Thanks for your attention