Differential Equations

Lecture 18

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Page 172-5 Find a series solution of Airy's equation about $x_0 = 0$:

$$y'' - xy = 0$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting these expressions into the equation, we obtain

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^{n} - \sum_{n=1}^{\infty} a_{n-1} x^{n} = 0$$

$$2\times 1\times a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_{n-1}\right] x^{n} = 0$$

For this equation to be valid for all *x*, the coefficient of each power of *x* must be zero; hence $a_2 = 0$ and

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}, \quad n = 1, 2, 3, \ldots$$

$$a_{n+3} = \frac{a_{n}}{(n+3)(n+2)} \quad n = 0,1,2,3,\ldots$$

We have $a_2 = 0$ **and** $a_{n+3} = \dfrac{a_n}{(n+2)(n+3)} \quad n = 0,1,2,3,\ldots$

$$n = 0 \rightarrow a_3 = \frac{a_0}{2\times 3}, \qquad n = 1 \rightarrow a_4 = \frac{a_1}{3\times 4}$$

$$n = 2 \rightarrow a_5 = \frac{a_2}{4\times 5} = 0$$

$$n = 3 \rightarrow a_6 = \frac{a_3}{5\times 6} = \frac{a_0}{(2\times 3)(5\times 6)},$$

$$n = 4 \rightarrow a_7 = \frac{a_4}{6\times 7} = \frac{a_1}{(3\times 4)(6\times 7)},$$

$$n = 5 \rightarrow a_8 = \frac{a_5}{7\times 8} = 0$$

$$n = 6 \to a_9 = \frac{a_6}{8\times 9} = \frac{a_0}{(2\times 3)(5\times 6)(8\times 9)}$$

$$n = 7 \to a_{10} = \frac{a_7}{9\times 10} = \frac{a_1}{(3\times 4)(6\times 7)(9\times 10)}, \cdots$$

The hardest part, as usual, is to recognize the patterns evolving; in this case we have to consider three cases:

1) All the terms $a_2, a_5, a_8, \ldots$ are equal to zero. We can write this in compact form as

$$a_{3n+2} = 0, \qquad \text{n} = 0,1,2,3,\ldots$$

2) All the terms $a_3, a_6, a_9, \ldots$ are multiples of a_0. We can be more precise:

$$a_{3n} = \frac{a_0}{(2\times 3)(5\times 6)\cdots((3n-1)(3n))}, \; n = 1, 2, \ldots$$

3) All the terms $a_4, a_7, a_{10}, \ldots$ are multiples of a_1. We can be more precise:

$$a_{3n+1} = \frac{a_1}{(3\times 4)(6\times 7)\cdots((3n)(3n+1))},\ n = 1, 2, \ldots$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \sum_{n=3}^{\infty} a_n x^n$$

$$y(x) = a_0\left[1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{(2\times 3)(5\times 6)\cdots((3n-1)(3n))}\right] +$$

$$a_1\left[x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{(3\times 4)(6\times 7)\cdots((3n)(3n+1))}\right]$$

where a_0, a_1 are arbitrary (determined by initial conditions). Consider the two cases

$$(1)\quad a_0 = 1,\ a_1 = 0 \quad \Leftrightarrow \quad y(0) = 1,\ y'(0) = 0$$

$$(2)\quad a_0 = 0,\ a_1 = 1 \quad \Leftrightarrow \quad y(0) = 0,\ y'(0) = 1$$

The corresponding solutions $y_1(x)$, $y_2(x)$ are linearly independent, since $W(y_1, y_2)(0) = 1 \neq 0$, where

$$W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = y_1(0)\,y_2'(0) - y_1'(0)\,y_2(0) = 1$$

the corresponding solutions $y_1(x)$, $y_2(x)$ are linearly independent, and thus are fundamental solutions for Airy's equation, with general solution

$$y(x) = c_1\, y_1(x) + c_1\, y_2(x)$$

Thus given the initial conditions

$$y(0) = a_0, \qquad y'(0) = a_1$$

the solutions are, respectively,

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{(2\times 3)(5\times 6)\cdots((3n-1)(3n))},$$

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{(3\times 4)(6\times 7)\cdots((3n)(3n+1))}$$

Find a series solution of Airy's equation about x_0 = 1:

$$y'' - xy = 0, \; -\infty < x < \infty$$

Thus every point x is an ordinary point. We will take $x_0 = 1$. Assuming a series solution and differentiating, we obtain

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n, \qquad y'(x) = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1},$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

Substituting these into ODE & shifting indices, we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n = x \sum_{n=0}^{\infty} a_n (x-1)^n$$

Our equation is

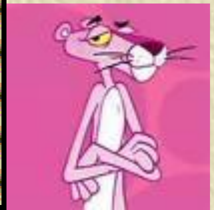
$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n = x\sum_{n=0}^{\infty}a_n(x-1)^n$$

The *x* on right side can be written as 1 + (*x* – 1); and thus

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n = [1+(x-1)]\sum_{n=0}^{\infty}a_n(x-1)^n$$

$$= \sum_{n=0}^{\infty}a_n(x-1)^n + \sum_{n=0}^{\infty}a_n(x-1)^{n+1}$$

$$= \sum_{n=0}^{\infty}a_n(x-1)^n + \sum_{n=1}^{\infty}a_{n-1}(x-1)^n$$



Thus our equation becomes

$$2a_2+\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n=a_0+\sum_{n=1}^{\infty}a_n(x-1)^n+\sum_{n=1}^{\infty}a_{n-1}(x-1)^n$$

Thus the recurrence relation is

$$(n+2)(n+1)a_{n+2}=a_n+a_{n-1},\ (n\geq 1)$$

Equating like powers of *x* -1, we obtain

$$2a_2=a_0 \qquad \Rightarrow a_2=\frac{a_0}{2},$$

$$n=1\rightarrow 3\times 2a_3=a_1+a_0 \qquad \Rightarrow a_3=\frac{a_0}{6}+\frac{a_1}{6},$$

$$n=2\rightarrow 4\times 3a_4=a_2+a_1 \qquad \Rightarrow a_4=\frac{a_0}{24}+\frac{a_1}{12},$$

$$\vdots$$

$a_0 = \text{arbitrary}$

$a_1 = \text{arbitrary}$

$a_2 = \frac{a_0}{2},$

$a_3 = \frac{a_0}{6} + \frac{a_1}{6},$

$a_4 = \frac{a_0}{24} + \frac{a_1}{12},$

$\vdots$

We now have the following information:

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$y(x) = a_0\left[1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \cdots\right]$$

$$+ a_1\left[(x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \cdots\right]$$

$$y(x) = a_0 y_3(x) + a_1 y_4(x)$$

The recursion has three terms,

$$(n+2)(n+1)a_{n+2} = a_n + a_{n-1},\ (n \geq 1)$$

and determining a general formula for the coefficients a_n can be difficult or impossible.

However, we can generate as many coefficients as we like, preferably with the help of a computer algebra system.