Differential Equations

Lecture 17

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In mathematics, a power series (in one variable) is an infinite series of the form

$$f(x)=\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$f'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$$

$$f''(x)=\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 3\times 2a_3 x + 4\times 3a_4 x^2 \ldots$$

And so on, and each of the resulting series converges for $|x| < R$

$$g(x)=\sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots$$

$$f(x)\pm g(x)=\sum_{n=0}^{\infty} (a_n \pm b_n) x^n = (a_0 \pm b_0) + (a_1 \pm b_1)x$$

$$+(a_2 \pm b_2)x^2 \ldots$$

$$f(x)g(x)=\sum_{n=0}^{\infty} c_n x^n \quad c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$$

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} a_{n-k} b_k$$

$$c_2 = \sum_{k=0}^{2} a_k b_{2-k} = a_0 b_2 + a_1 b_1 + a_2 b_0$$

Series Equality

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n \qquad a_0 = b_0, a_1 = b_1, \ldots$$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = 0$$

For | *x*| < *R*

$$a_0 = 0, a_1 = 0, \ldots$$

Taylor Series

Suppose that $\Sigma\, a_n x^n$ converges to $f(x)$ for $|x| < R$. Then the value of a_n is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + f^{(n)}(x_0)\frac{(x-x_0)^n}{n!} + \ldots$$

$$a_n = \frac{f^{(n)}(x_0)}{n!},$$

and the series is called the Taylor series for f about x_0.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots$$

Maclaurin series: the series is called the Taylor series for f about $x = 0$.

Example – Taylor Series

Example: Find the Taylor series for:

$$f(x) = \ln(x), \qquad x_0 = 1$$

Find the value of the function and derivatives for the function at x_0=1

$$f(x)=\ln(x) \quad \Rightarrow f(x_0)=\ln(1)=0$$

$$f'(x)=\frac{1}{x} \quad \Rightarrow f'(x_0)=\frac{1}{1}=1$$

$$f''(x)=-\frac{1}{x^2} \quad \Rightarrow f''(x_0)=-\frac{1}{1^2}=-1$$

$$f'''(x)=\frac{2}{x^3} \quad \Rightarrow f'''(x_0)=\frac{2}{1^3}=2$$

$$\vdots$$

$$f^{(n)}(x)=\frac{(n-1)!(-1)^{n-1}}{x^n}$$

$$\Rightarrow f^{(n)}(x_0)=\frac{(n-1)!(-1)^{n-1}}{1^n}=(n-1)!(-1)^{n-1}$$

Use the Taylor series formula:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2!}$$

$$+ \ldots + f^{(n)}(x_0)\frac{(x - x_0)^n}{n!} + \ldots$$

$$\Rightarrow \ln(x) = 0 + (x-1) - \frac{(x-1)^2}{2!} + \frac{2!(x-1)^3}{3!}$$

$$+ \ldots + (n-1)!(-1)^{n-1}\frac{(x-1)^n}{n!} + \ldots$$

$$\Rightarrow \ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$$

$$+ \ldots + (-1)^{n-1}\frac{(x-1)^n}{n} + \ldots$$

Example:Find the Maclaurin series of the function $f(x)=e^x$ and radius of convergence.

$$f(x)=e^x \qquad f^{(n)}(x)=e^x \qquad f^{(n)}(0)=e^0=1$$

Taylor series for *f* at 0 (that is, the Maclaurin series) is

$$e^x=\sum_{x=0}^{\infty}\frac{f^{(x)}(0)}{n!}x^n=\sum_{x=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

To find the radius of convergence we let $a_n=x^n/n!$. then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|=\lim_{n\to\infty}\frac{|x|}{n+1}\to 0<1$$

So, by the Ratio Test, the series converges for all x and the radius of convergence is $R=\infty$.

Example: Find the Maclaurin series for *sin x*

$$f(x) = \sin x \qquad f(0) = 0$$

$$f'(x) = \cos x \qquad f'(0) = 1$$

$$f''(x) = -\sin x \qquad f''(0) = 0$$

$$f'''(x) = -\cos x \qquad f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$$

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Example: Find the Maclaurin series for *cos x*

$$\cos x = \frac{d}{dx}(\sin x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$$

$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \ldots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Euler Formula

$$e^{ix} = \cos x + i \sin x$$

Proof:

Maclaurin Series

$$\begin{cases} e^{x} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \cos(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ \sin(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \end{cases}$$

$$e^{ix} = \cos x + i \sin x$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= \cos(x) + i \sin(x)$$

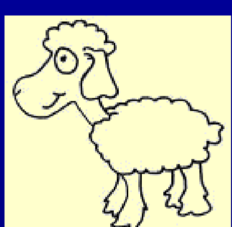
$$e^{i\pi} = -1$$

Series solutions of first order equation

$$y' = y \qquad \frac{dy}{dx} = y \qquad \rightarrow \frac{dy}{y} = dx$$

$$\ln y = x + \ln c \rightarrow \ln \frac{y}{c} = x$$

$$y = ce^{x}$$



$$y=\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \ldots n a_n x^{n-1} +$$

$$(n+1)a_{n+1}x^n + \ldots$$

Since y’=y, the above series must have the same coefficients:

$$x^0 \rightarrow a_1 = a_0, \qquad (n+1)a_{n+1} = a_n$$

$$x^1 \rightarrow 2a_2 = a_1 \qquad \vdots$$

$$x^2 \rightarrow 3a_3 = a_2$$

These equations enable us to express each a_n in terms of a_0:

$$a_1 = a_0, \qquad a_2 = \frac{a_1}{2} = \frac{a_0}{2} \qquad a_3 = \frac{a_2}{3} = \frac{a_0}{2 \times 3}$$

$$a_n = \frac{a_0}{n!}$$

When these coefficient are inserted in below equation, we obtain our power series solution

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

$$y = a_0 + a_0 x + \frac{a_0}{2!} x^2 + \ldots + \frac{a_0}{n!} x^n + \ldots$$

$$y = a_0 \left(1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots\right) = a_0 e^x$$

Example: As an illustration we consider the function

$$y = (1+x)^p \ *$$ **Where p is an arbitrary constant.**

It is easy to see that * is the indicated particular solution of the following differential equation:

$$(1+x)y' = py, \quad y(0) = 1$$

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \ldots n a_n x^{n-1} + (n+1)a_{n+1} x^n + \ldots$$

$$xy' = a_1 x + 2a_2 x^2 + 3a_3 x^3 + \ldots n a_n x^n + (n+1)a_{n+1} x^{n+1} + \ldots$$

$$y' = a_1 + 2a_2x + 3a_3x^2 + \dots na_nx^{n-1} + (n+1)a_{n+1}x^n + \dots$$

$$xy' = a_1x + 2a_2x^2 + 3a_3x^3 + \dots na_nx^n + (n+1)a_{n+1}x^{n+1} + \dots$$

$$py = pa_0 + pa_1x + pa_2x^2 + \dots + pa_nx^n + \dots$$

$$(1+x)y' = py$$

$$a_1 = pa_0 \qquad 2a_2 + a_1 = pa_1 \qquad 3a_3 + 2a_2 = pa_2$$

$$\dots(n+1)a_{n+1} + na_n = pa_n, \dots$$

The initial condition implies that *a_0=1* so

$$a_1 = p \qquad a_2 = \frac{a_1(p-1)}{2} = \frac{p(p-1)}{2}$$

$$a_3 = \frac{a_2(p-2)}{3} = \frac{p(p-1)(p-2)}{2\times 3}, \ldots$$

$$a_n = \frac{p(p-1)(p-2)\ldots(p-n+1)}{n!}$$

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

$$y = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots$$

$$+ \frac{p(p-1)(p-2)\ldots(p-n+1)}{n!}x^n + \ldots$$

$$y = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots$$

$$+ \frac{p(p-1)(p-2)\ldots(p-n+1)}{n!}x^n + \ldots$$

$$y = (1+x)^p$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots$$

$$+ \frac{p(p-1)(p-2)\ldots(p-n+1)}{n!}x^n + \ldots$$

This result is called the binomial series, and generalizes the binomial theorem to the case of an arbitrary exponent.

Second order linear equations. Ordinary points

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

we want to solve the equation below in a neighborhood of a point of interest x_0:

The point x_0 is called an ordinary point if $P(x_0) \neq 0$. Since P is continuous, $P(x) \neq 0$ for all x in some interval about x_0. Since p and q are continuous. Theorem 3.2.1 says there is a unique solution, given initial conditions $y(x_0) = y_0$, $y'(x_0) = y_0'$

Find a series solution of the equation $y'' + y = 0$

These function are analytic at all points, so we seek a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting these expressions into the equation, we obtain

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Index shifting

Consider the example

$$\int_2^5 (x+1)^5 \, dx.$$

Using a simple substitution *u*=*x*+1, we can rewrite this integral as

$$\int_3^6 u^5 \, du,$$

$$\int_2^5 (x+1)^5 \, dx = \int_3^6 x^5 \, dx.$$

The expression (*x*+1) is "shifted down" by one unit to *x*, while the limits of integration are "shifted up" by one unit from 2 to 3, and 5 to 6.

Summation is just a special case of integration, so an analogous "index shifting" will work:

$$\sum_{2}^{5}(n+1)^5 = \sum_{3}^{6} n^5.$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$x^{k-2} \rightarrow x^n$$

$$k-2=n \rightarrow$$

$$k=n+2$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right] x^n = 0$$

For this equation to be valid for all *x*, the coefficient of each power of *x* must be zero, and hence

$$(n+2)(n+1)a_{n+2} + a_n = 0, \quad n = 0, 1, 2, \ldots$$

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}, \quad n = 0, 1, 2, \ldots$$ **recurrence relation**

Next, we find the individual coefficients a_0, a_1, a_2, …

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}$$

$$n = 0 \rightarrow a_2 = -\frac{a_0}{2 \times 1},$$

$$n = 2 \rightarrow a_4 = -\frac{a_2}{4 \times 3} = \frac{a_0}{4 \times 3 \times 2 \times 1},$$

$$n = 4 \rightarrow a_6 = -\frac{a_4}{6 \times 5} = -\frac{a_0}{6 \times 5 \times 4 \times 3 \times 2 \times 1},$$

$$a_{2k} = \frac{(-1)^k a_0}{(2k)!}, \quad k = 1, 2, 3, \ldots$$

Odd Coefficients

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}$$

$$n = 1 \rightarrow a_3 = \frac{-a_1}{3 \times 2}$$

$$n = 3 \rightarrow a_5 = -\frac{a_3}{5 \times 4} = \frac{a_1}{5 \times 4 \times 3 \times 2 \times 1},$$

$$n = 5 \rightarrow a_7 = -\frac{a_5}{7 \times 6} = -\frac{a_1}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1},$$

$$\vdots$$

$$a_{2k+1} = \frac{(-1)^k a_1}{(2k+1)!}, \quad k = 1, 2, 3, \ldots$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

where $a_{2k} = \dfrac{(-1)^k}{(2k)!} a_0, \quad a_{2k+1} = \dfrac{(-1)^k}{(2k+1)!} a_1$

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Note: a_0 and a_1 are determined by the initial conditions.

$$y(x) = a_0 y_1 + a_1 y_2$$

$$y(x) = a_0 \cos x + a_1 \sin x$$