Differential Equations

Lecture 16

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Differential Equation Solution using Series

A series is often represented as the sum of a sequence of terms, for example this arithmetic sequence:

1 + 2 + 3 + 4 + 5 + ... + 99 + 100

Thus, the general form of a geometric sequence is

$$\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\ldots=(\frac{1}{2})^{n}$$

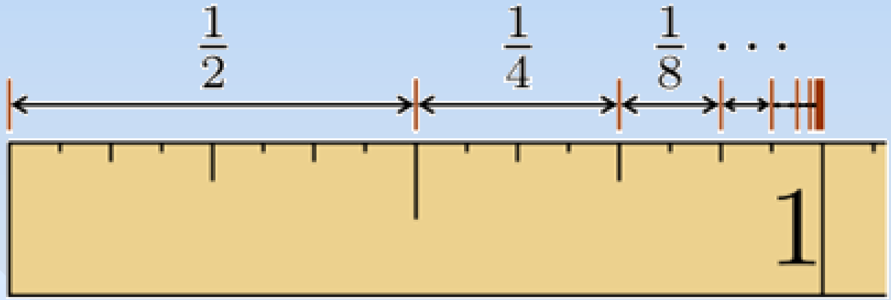
In mathematics, a geometric series is a series with a constant ratio between successive terms. For example, the series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$$

is geometric, because each term is equal to half of the previous term

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots=1$$

The sum of this series is 1, as illustrated in the following picture:



Geometric series are the simplest examples of infinite series with finite sums.

$$s = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$$

$$\frac{1}{2}s = \quad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$$

$$s - \frac{1}{2}s = 1 \to s = 2$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots = 2$$

One also says that this series converges to 2.

$$s = a + ax + ax^2 + ax^3 + \dots$$

$$xs = \quad ax + ax^2 + ax^3 + \dots$$

$$s - xs = a \to s = \frac{a}{1-x}$$

There is a general formula for the sum of a geometric series:

Here *a* is the first term of the series, and *x* is the common ratio. When *a* = 1, this specializes to the important formula

$$1 + x + x^2 + x^3 + \ldots + x^n + \ldots = \frac{1}{1-x} \quad \textbf{for } |x| < 1.$$

Keep in mind that this formula only works when the series converges (i.e. when the magnitude of *x* is less than one).

Definition: **An infinite series of the form**

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

is called a *power series* in *x*. where the a_n are real numbers, and *x* is a real variable.

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots$$

This series is a power series in x-x_0

Convergent Power Series

A power series about the point *x* has the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots *$$

and is said to converge at a point *x* if the limit

$$\lim_{m\to\infty} \sum_{n=0}^{m} a_n x^n$$

exists, and in this case the sum of the series is the value of this limit. It is obvious that * always converges at the point x=0

With respect to the arrangement of their points of convergence, all power series in *x* fall into one of another of three major categories.

1) The geometric series $$\sum_{n=0}^{\infty} n!\,x^n = 1 + x + 2!\,x^2 + 3!\,x^3 + \ldots$$

is another example of a power series . This series is converges only when *x* = 0;

2) $$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

This series is converges absolutely for all *x*

3) $$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$$

for | *x*| < 1 is converges and | *x*| > 1 is divergence .

Some power series in x behave like (1) and converge only for x=0. These are of no interest to us.

Each power series in *x* has a radius of convergence *R*, where *0*<*R*<∞ such that $\sum x^n$ converges absolutely for all *x* with | *x*| < *R*, and diverges for all *x* with | *x*| > *R*.

In many important cases the value of R can be found as follows.

$$\sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \ldots$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = L < 1 \rightarrow \text{the series is converges}$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = L > 1 \rightarrow \text{the series is diverges}$$

Example: Find the kind of series:

$$1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}+\ldots$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{n!}{(n+1)!}=\lim_{n\to\infty}\frac{1}{n+1}=0<1$$

convergence

Example: Find the kind of series:

$$\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\ldots+\frac{2n-1}{2^n}+\ldots$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(2n+1)}{2^{n+1}}\frac{2^n}{(2n-1)}$$

$$=\lim_{n\to\infty}\frac{1}{2}\cdot\frac{1+\frac{1}{2n}}{1-\frac{1}{2n}}=\frac{1}{2}<1$$

convergence

Ratio Test

One of the most useful tests for the absolute convergence of a power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is the ratio test. If $a_n \neq 0$, and if, for a fixed value of x,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||x| = L,$$

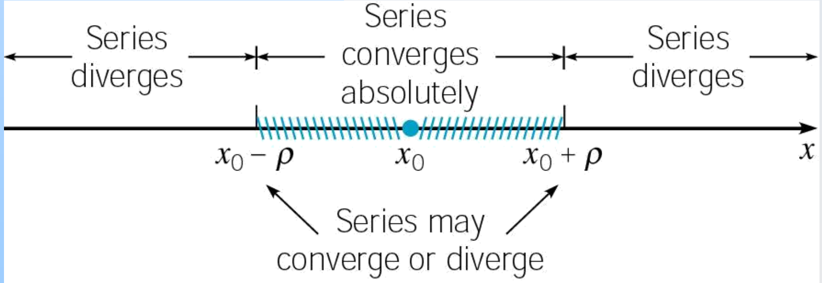
then the power series converges absolutely at that value of x if $L < 1$ and diverges if $L > 1$. The test is inconclusive if $L = 1$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||x| < 1 \qquad |x| < \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

Radius of Convergence

There is a nonnegative number *R*, called the radius of convergence, such that $\Sigma\, a_n x^n$ converges absolutely for all *x* satisfying $|x| < R$ and diverges for $|x| > R$.



The series may either converge or diverge when $|x| = R$.

For a series that converges only at x_0, we define R to be zero.

For a series that converges for all x, we say that R is infinite.

If $R > 0$, then $|x - x_0| < R$ is called the interval of convergence.

Definition: The number R described above is called the radius of convergence of the power series. By allowing $R = 0$ and $R = \infty$, we can consider *every* power series to have a radius of convergence.

Thus every power series has a radius of convergence. We sometimes call the interval $(-R, R)$, where the power series is guaranteed to converge, the interval of convergence. It is characterised by the fact that the series converges (absolutely) inside this interval and diverges outside the interval.

Examples (Calculation of R)

$$\sum \frac{nx^n}{2^{n+1}}$$

$$\frac{|(n+1)term|}{|(n)term|} = \frac{(n+1)\left|x^{n+1}\right|}{2^{n+2}} \frac{2^{n+1}}{n\left|x^n\right|} = \frac{(n+1)|x|}{2n}$$

$$\lim_{n\to\infty} \frac{(n+1)|x|}{2n} \to \frac{|x|}{2}$$

Hence $R = 2$

The given series converges for $|x| < 2$

The given series diverges for $|x| > 2$

Radius of Convergence

Find the radius of convergence of $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!\,x^n}{(n!)^2}$.

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{a\to\infty}\frac{(2n)!/(n!)^2}{(2(n+1))!/((n+1)!)^2}$$

$$=\lim_{n\to\infty}\frac{((n+1)!)^2}{(n!)^2}\,\frac{(2n)!}{(2n+2)!}=\lim_{n\to\infty}\left(\frac{(n+1)!}{n!}\right)^2\frac{(2n)!}{(2n+2)!}$$

$$=\lim_{n\to\infty}\left(\frac{1\times 2\times 3\times\ldots\times n\times(n+1)}{1\times 2\times 3\times\ldots\times n}\right)^2\frac{1\times 2\times 3\times\ldots\times(2n)}{1\times 2\times 3\times\ldots\times(2n)\times(2n+1)\times(2n+2)}$$

$$\lim_{n\to\infty}(n+1)^2\frac{1}{(2n+2)(2n+1)}=\lim_{n\to\infty}\frac{n^2+2n+1}{4n^2+5n+2}=\lim_{n\to\infty}\frac{1+2/n+1/n^2}{4+5/n+2/n^2}=\frac{1}{4}$$

Find the radius of convergence for the power series below.

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{n!}$$

Using the ratio test, we obtain $\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right|$

$$\lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(n+1)!}\,\frac{n!}{(x+2)^n}\right| = |x+2|\lim_{n\to\infty}\left|\frac{n!}{(n+1)n!}\right| = 0 < 1$$

$$\text{for} \quad -\infty < x < \infty$$

Thus the interval of convergence is (-∞, ∞), and hence the radius of convergence is infinite.

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Use the ratio test to verify that *R=0* for the following series:

$$\sum_{n=0}^{\infty} n!\,x^n = 1 + x + 2!\,x^2 + 3!\,x^3 + \ldots$$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{1}{n+1}\right| = 0$$

In mathematics, a power series (in one variable) is an infinite series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 3\times 2a_3 x + 4\times 3a_4 x^2 \ldots$$

And so on, and each of the resulting series converges for $|x| < R$

$$a_n = \frac{f^{(n)}(0)}{n!}$$

These successive differentiated series yield the basic formula linking the a_n's to f(x) and its dervatives.

$$g(x)=\sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots$$

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n = (a_0 \pm b_0) + (a_1 \pm b_1) x$$

$$+ (a_2 \pm b_2) x^2 \ldots$$

$$f(x) g(x) = \sum_{n=0}^{\infty} c_n x^n \quad c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$$

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} a_{n-k} b_k$$

$$c_2 = \sum_{k=0}^{2} a_k b_{2-k} = a_0 b_2 + a_1 b_1 + a_2 b_0$$

Series Equality

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n \qquad a_0 = b_0, a_1 = b_1, \ldots$$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = 0$$

For | *x*| < *R*

$$a_0 = 0, a_1 = 0, \ldots$$

Taylor Series

Suppose that Σ $a_n x^n$ converges to *f* (*x*) for |*x* | < R. Then the value of a_n is given by

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+$$

$$\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots+f^{(n)}(x_0)\frac{(x-x_0)^n}{n!}+\ldots$$

$$a_n=\frac{f^{(n)}(x_0)}{n!},$$

and the series is called the Taylor series for *f* about x_0 .

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots$$

Maclaurin series: the series is called the Taylor series for *f* about *x* = 0.

Example – Taylor Series

Example: Find the Taylor series for:

$$f(x) = \ln(x), \qquad x_0 = 1$$

Find the value of the function and derivatives for the function at $x_0=1$

$$f(x) = \ln(x) \quad \Rightarrow f(x_0) = \ln(1) = 0$$

$$f'(x) = \frac{1}{x} \quad \Rightarrow f'(x_0) = \frac{1}{1} = 1$$

$$f''(x) = -\frac{1}{x^2} \quad \Rightarrow f''(x_0) = -\frac{1}{1^2} = -1$$

$$f'''(x) = \frac{2}{x^3} \quad \Rightarrow f'''(x_0) = \frac{2}{1^3} = 2$$

$$\vdots$$

$$f^{(n)}(x) = \frac{(n-1)!(-1)^{n-1}}{x^n}$$

$$\Rightarrow f^{(n)}(x_0) = \frac{(n-1)!(-1)^{n-1}}{1^n} = (n-1)!(-1)^{n-1}$$

Use the Taylor series formula:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!}$$

$$+\ldots+ f^{(n)}(x_0)\frac{(x-x_0)^n}{n!} + \ldots$$

$$\Rightarrow \ln(x) = 0 + (x-1) - \frac{(x-1)^2}{2!} + \frac{2!(x-1)^3}{3!}$$

$$+\ldots+(n-1)!(-1)^{n-1}\frac{(x-1)^n}{n!} + \ldots$$

$$\Rightarrow \ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$$

$$+\ldots+(-1)^{n-1}\frac{(x-1)^n}{n} + \ldots$$

Example:Find the Maclaurin series of the function *f(x)=e^x* and radius of convergence.

$$f(x)=e^{x} \qquad f^{(n)}(x)=e^{x} \qquad f^{(n)}(0)=e^{0}=1$$

$$e^{x}=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^{n}=\sum_{n=0}^{\infty}\frac{x^{n}}{n!}=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\dots$$

To find the radius of convergence we let $a_n=x^n/n!$. then

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^{n}}\right|=\lim_{n\to\infty}\frac{|x|}{n+1}\to 0<1$$

So, by the Ratio Test, the series converges for all x and the radius of convergence is $R=\infty$.

Example: Find the Maclaurin series for *sinx*

$$f(x) = \sin x \qquad f(0) = 0$$

$$f'(x) = \cos x \qquad f'(0) = 1$$

$$f''(x) = -\sin x \qquad f''(0) = 0$$

$$f'''(x) = -\cos x \qquad f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$$

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Example: Find the Maclaurin series for *cosx*

$$\cos x = \frac{d}{dx}(\sin x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$$

$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \ldots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Euler Formula

$$e^{ix} = \cos x + i \sin x$$

Proof:

Maclaurin Series

$$\begin{cases} e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \dfrac{x^n}{n!} \\ \cos(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n)!} \\ \sin(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!} \end{cases}$$

$$e^{ix} = \cos x + i \sin x$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= \cos(x) + i \sin(x)$$

$$e^{i\pi} = -1$$

Analytic Functions

- A function f that has a Taylor series expansion about $x = x_0$

$$f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

with a radius of convergence $R > 0$, is said to be analytic at x_0.

- All of the familiar functions of calculus are analytic.
- For example, sin x and e^x are analytic everywhere, while $1/x$ is analytic except at $x = 0$, and tan x is analytic except at odd multiples of $\pi/2$.
- If f and g are analytic at x_0, then so are $f \pm g$, fg, and f/g ;

A real function is said to be analytic if it possesses derivatives of all orders and agrees with its Taylor series in a neighborhood of every point.

$$f(x)=\frac{1}{x-1}, \qquad x_0=0 \qquad f(x_0)=-1$$

Analytic Functions

$$f(x)=\frac{1}{x}, \qquad x_0=0$$

Non-analytic Functions