Differential Equations

Lecture 14

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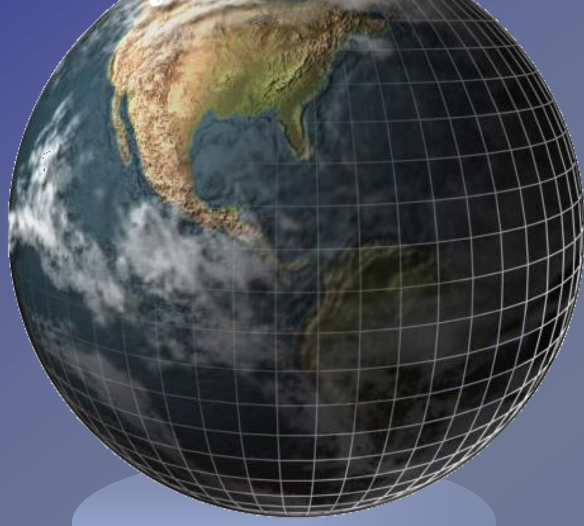
Applications of 2nd Order Linear

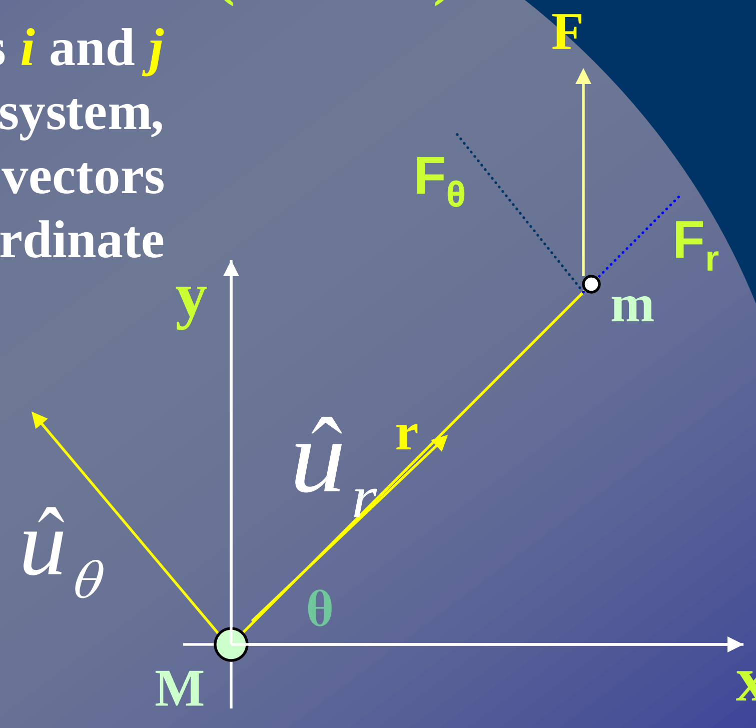
Newton’s Law of Gravitation and the Motion of the Planets

Our purpose in this section is to deduce Kepler’s laws of planetary motion from Newton’s law of universal gravitation.



The motion of a small particle of mass *m* (a planet) under the attraction of a fixed large particle of mass *M* (the sun).

Just as we have two distinguished unit vectors *i* and *j* corresponding to the Cartesian coordinate system, we can likewise define two distinguished unit vectors u_r and u_θ corresponding to the polar coordinate system:



$$\hat{u}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{u}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\vec{r} = x\hat{i} + y\,\hat{j} = r\cos\theta\,\hat{i} + r\sin\theta\hat{j}$$

$$= r[\cos\,\hat{i} + \sin\theta\,\hat{j}] = r\,\hat{u}_r$$

$$\hat{u}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \qquad \hat{u}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\frac{d\hat{u}_r}{d\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = \hat{u}_\theta$$

$$\frac{d\hat{u}_\theta}{d\theta} = -\cos\theta\,\hat{i} - \sin\theta\,\hat{j} = -\hat{u}_r$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(r\hat{u}_r\right) = \frac{dr}{dt}\hat{u}_r + r\frac{d\hat{u}_r}{dt}$$

$$= \frac{dr}{dt}\hat{u}_r + r\frac{d\hat{u}_r}{d\theta}\frac{d\theta}{dt} = \frac{dr}{dt}\hat{u}_r + r\frac{d\theta}{dt}\hat{u}_\theta$$

$$\vec{a}(t)=\frac{d\vec{v}}{dt}=\frac{d}{dt}\left(\frac{dr}{dt}\hat{u}_r+r\frac{d\theta}{dt}\hat{u}_\theta\right)$$

$$=\frac{d^2r}{dt^2}\hat{u}_r+\frac{dr}{dt}\frac{d\hat{u}_r}{dt}+\frac{dr}{dt}\frac{d\theta}{dt}\hat{u}_\theta+r\frac{d^2\theta}{dt^2}\hat{u}_\theta+r\frac{d\theta}{dt}\frac{d\hat{u}_\theta}{dt}$$

$$=\frac{d^2r}{dt^2}\hat{u}_r+\frac{dr}{dt}\frac{d\hat{u}_r}{d\theta}\frac{d\theta}{dt}+\frac{dr}{dt}\frac{d\theta}{dt}\hat{u}_\theta+r\frac{d^2\theta}{dt^2}\hat{u}_\theta+r\frac{d\theta}{dt}\frac{d\hat{u}_\theta}{d\theta}\frac{d\theta}{dt}$$

$$=\frac{d^2r}{dt^2}\hat{u}_r+\frac{dr}{dt}\hat{u}_\theta\frac{d\theta}{dt}+\frac{dr}{dt}\frac{d\theta}{dt}\hat{u}_\theta+r\frac{d^2\theta}{dt^2}\hat{u}_\theta+r\frac{d\theta}{dt}\left(-\hat{u}_r\right)\frac{d\theta}{dt}$$

$$=\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right]\hat{u}_r+\left[r\frac{d^2\theta}{dt^2}+2\frac{dr}{dt}\frac{d\theta}{dt}\right]\hat{u}_\theta$$

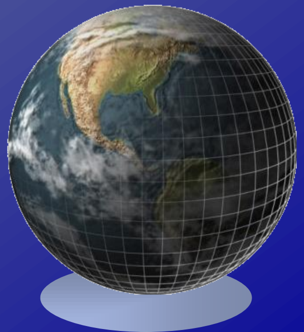
If the force *F* acting on *m* is written in the form:

$$\vec{F} = F_\theta \hat{u}_\theta + F_r \hat{u}_r$$

$$F_\theta = m\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)$$

$$F_r = m\left[\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]$$

These differential equations govern the motion of the particle m, and are valid redargless of the nature of the force.



Central Forces: F is called central force if it has no component perpendicular to r, if $F_\theta = 0$.

$$F_\theta = m\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right) = 0$$

$$r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = 0$$

On multiplying through by *r*, we obtain:

$$r^2\frac{d^2\theta}{dt^2} + 2r\frac{dr}{dt}\frac{d\theta}{dt} = 0$$

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0 \qquad r^2\frac{d\theta}{dt} = h$$

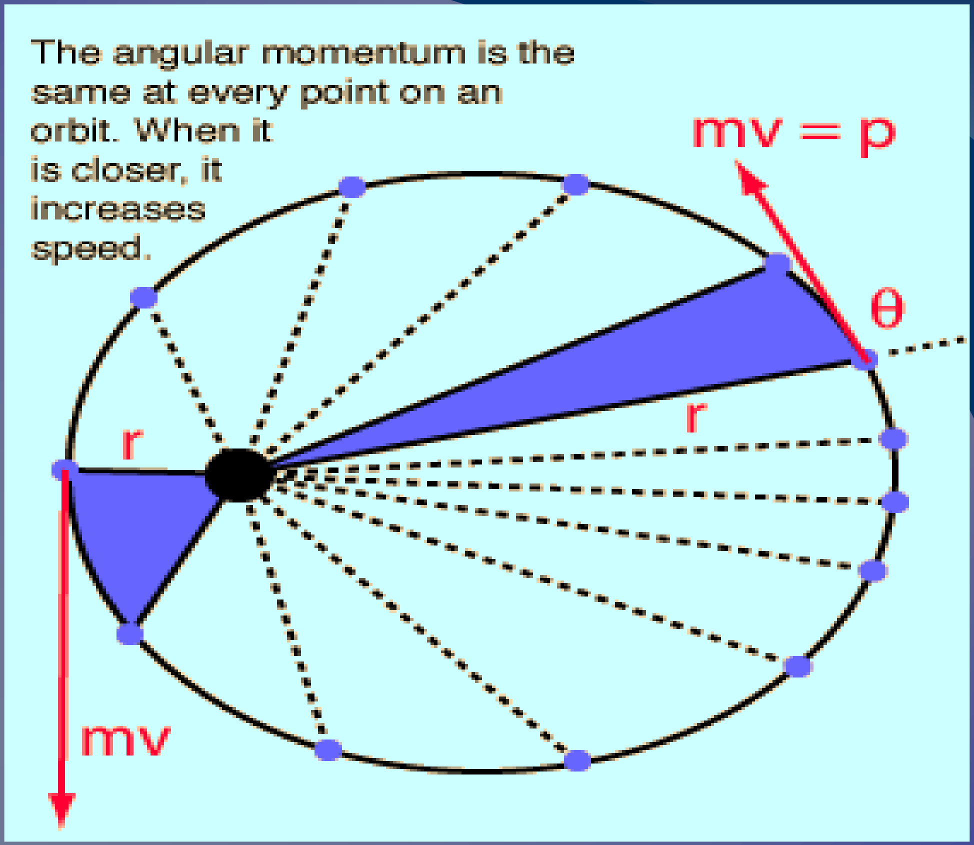
the specific relative angular momentum of the orbiting body

area constant

In astrodynamics, the specific relative angular momentum of an orbiting body with respect to a central body is the relative angular momentum of the first body per unit mass. Specific relative angular momentum plays a pivotal role in definition of orbit equations.

If the **area constant** h is zero, $d\theta/dt$ must vanish, i.e., θ must remain constant, so that the motion must take place on a straight line through the origin.

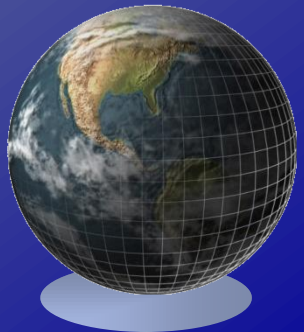
For an orbit, angular momentum is conserved, and this leads to one of Kepler's laws. For a circular orbit, L becomes



$$\vec{L} = \vec{r} \times m\vec{v}$$

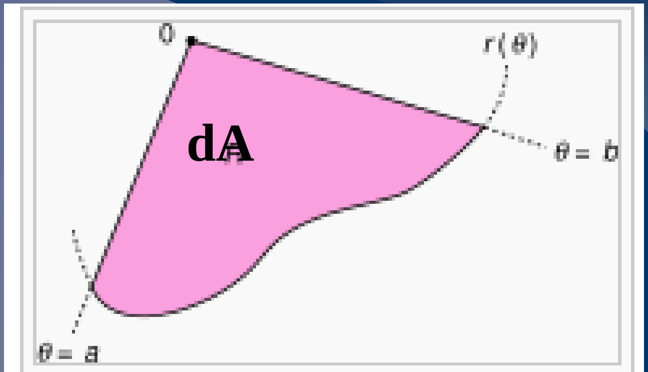
$$= r\hat{u}_r \times m(\frac{dr}{dt}\hat{u}_r + r\frac{d\theta}{dt}\hat{u}_\theta) = mr^2\frac{d\theta}{dt}\hat{k}$$

$$\vec{h} = \vec{L}/m \quad h = r^2\frac{d\theta}{dt}$$



Let *dA* denote the region enclosed by a curve *r*(θ) and the rays θ = *a* and θ = *b*, where $0 < b - a < 2\pi$. Then, the area of *dA* is

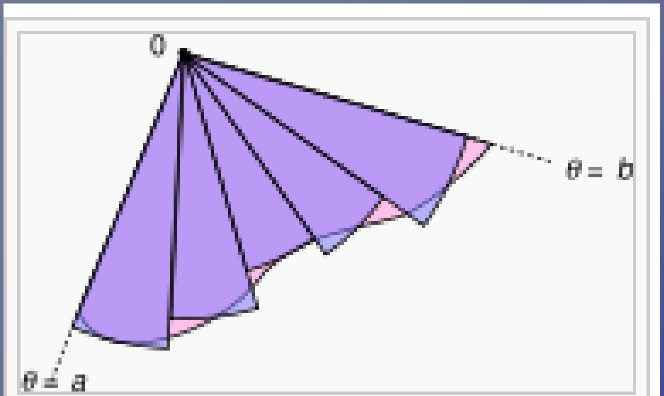
$$\frac{1}{2}\int_a^b r(\theta)^2 \, d\theta$$



For each subinterval $i = 1, 2, \ldots, n$, let θ_i be the midpoint of the subinterval, and construct a sector with the center at the pole, radius $r(\theta_i)$, central angle $\Delta\theta$ and arc length $r(\theta_i)\,\Delta\theta$.

The area of each constructed sector is

therefore equal to $\frac{1}{2} r(\theta_i)^2 \Delta\theta$. Hence, the total area of all of the sectors is



The region *R* is approximated by *n* sectors (here, *n* = 5).

$$\sum \frac{1}{2} r(\theta_i)^2 \, \Delta\theta$$

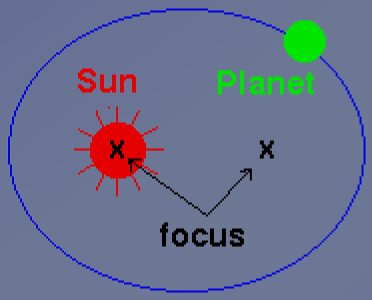
In the limit as $n \to \infty$, the sum becomes the Riemann sum for the above integral.

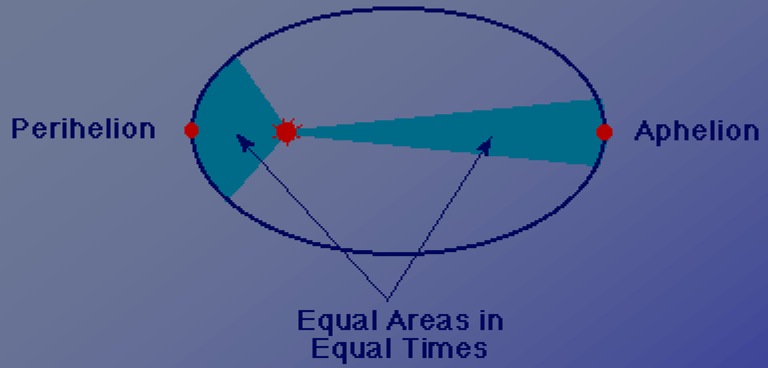
Recalling area differential in polar coordinates and abusing the notation,

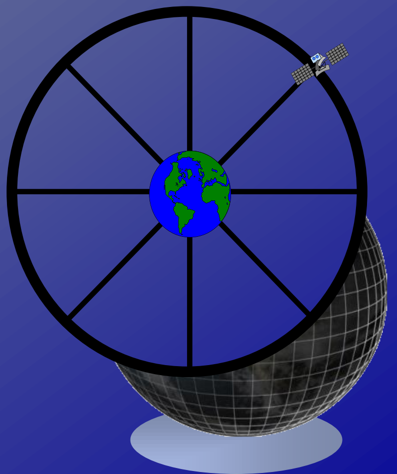
$$dA = \frac{1}{2} r^2 d\theta \qquad \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} h$$

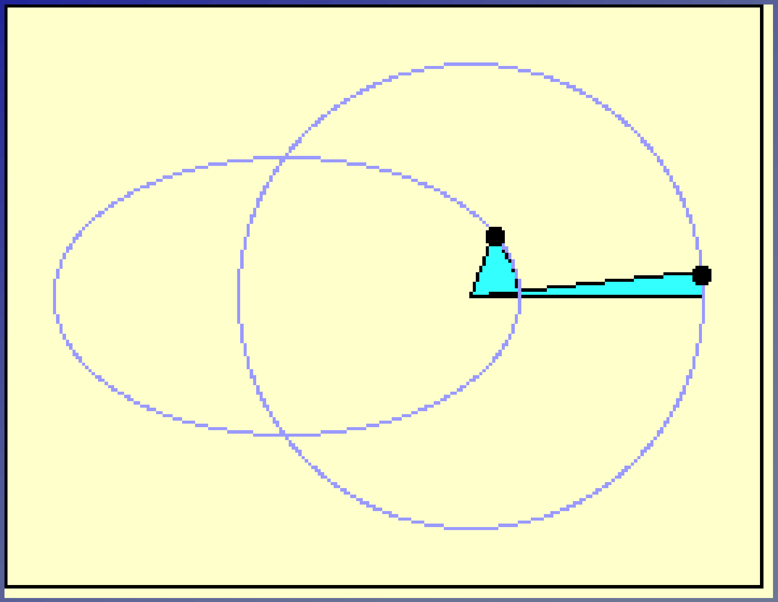
$$A(t_2) - A(t_1) = \frac{1}{2} h(t_2 - t_1)$$

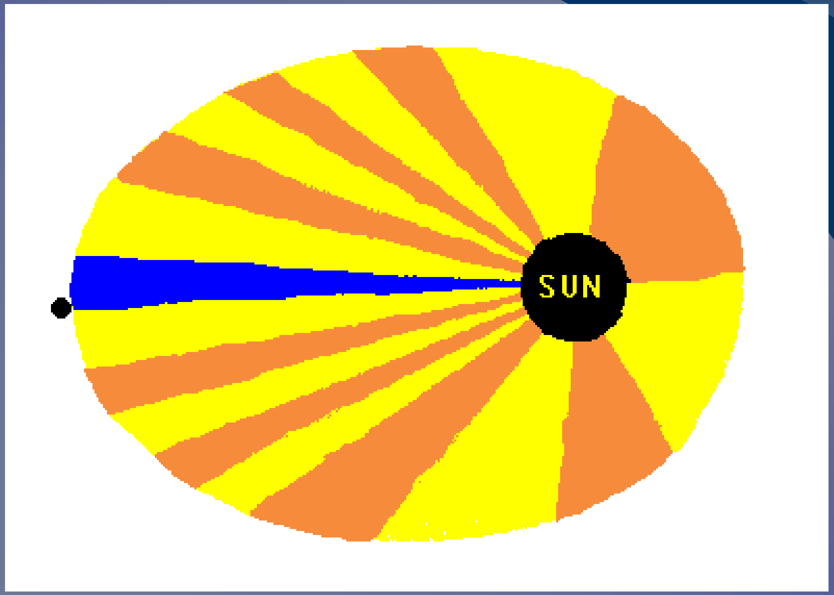
This yields Kepler's Second Law: **the radius vector *r* from the sun to a planet sweeps out equal areas in equal intervals of time.**











Planets move fastest when they are closest to the Sun

They move slowest when they are farthest from the Sun

Equal areas in equal times => the planet *must* be changing speed during its orbit

Kepler's First Law

Central gravitational forces

Orbital mechanics is the 'road map' to putting anything in Space

$$F_r = -G\frac{Mm}{r^2} \qquad k = GM$$

standard gravitational Parameter(km^3s^{-2})

$$F_r = -\frac{km}{r^2}$$

$$F_r = m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] \qquad \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{k}{r^2}$$

$$z = \frac{1}{r} \rightarrow \frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{z}\right) = -\frac{1}{z^2}\frac{dz}{dt} = -\frac{1}{z^2}\frac{dz}{d\theta}\frac{d\theta}{dt}$$

$$= -\frac{1}{z^2}\frac{dz}{d\theta}\frac{h}{r^2} = -h\frac{dz}{d\theta}$$

$$\frac{dr}{dt} = -h\frac{dz}{d\theta} \rightarrow \frac{d^2r}{dt^2} = -h\frac{d}{dt}\left(\frac{dz}{d\theta}\right)$$

$$= -h\frac{d}{d\theta}\left(\frac{dz}{d\theta}\right)\frac{d\theta}{dt} = -h\frac{d^2z}{d\theta^2}\frac{h}{r^2}$$

$$= -h^2z^2\frac{d^2z}{d\theta^2}$$

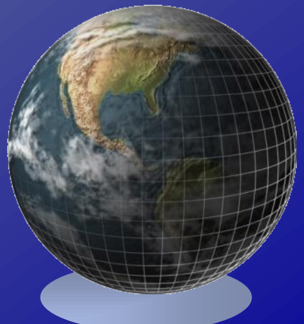
$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{k}{r^2}$$

$$\frac{d^2r}{dt^2} = -h^2z^2\frac{d^2z}{d\theta^2}$$

$$r = \frac{1}{z}$$

$$\frac{d\theta}{dt} = hz^2$$

$$-h^2z^2\frac{d^2z}{d\theta^2} - \frac{1}{z}h^2z^4 = -kz^2$$



$$\frac{d^2 z}{d\theta^2} + z = \frac{k}{h^2} \qquad \frac{d^2 z}{d\theta^2} + z = 0 \ \text{H.E.}$$

$$z = z_g + z_p \qquad z = A\sin\theta + B\cos\theta + \frac{k}{h^2} \quad \text{g.s.}$$

For the sake of simplicity, we shift the direction of the polar axis in such a way that *r* is minimal (that is, m is closet to the origin) when *θ=0*. This means that *z* is to be maximal in this direction, so

$$\frac{dz}{d\theta} = 0 \quad and \quad \frac{d^2 z}{d\theta^2} < 0 \qquad B > 0$$

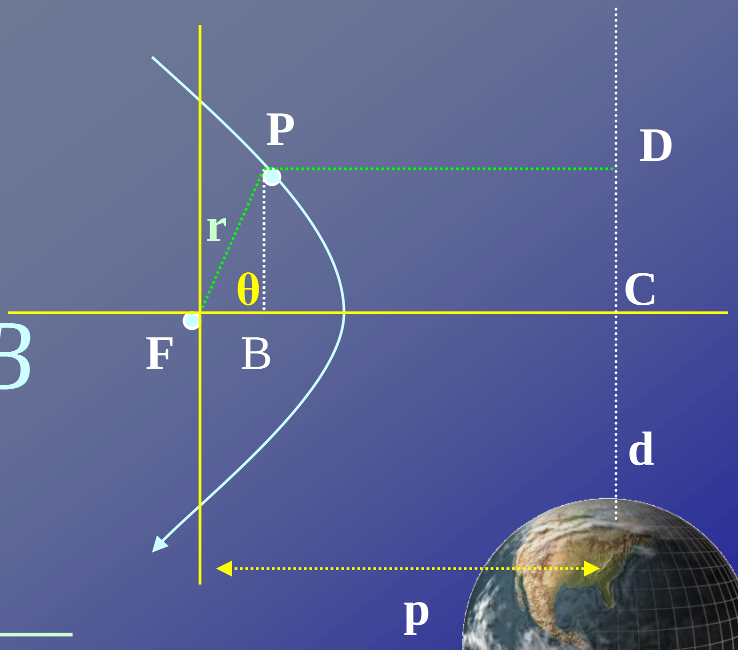
$$\frac{dz}{d\theta} = A\cos\theta - B\sin\theta \ \rightarrow A = 0$$

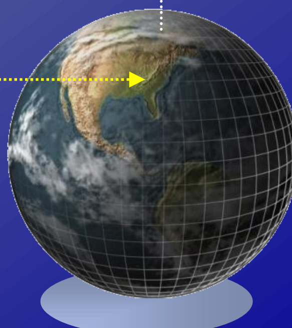
$$z = B\cos\theta + \frac{k}{h^2} \quad \frac{dz}{d\theta} = -B\sin\theta \quad \frac{d^2 z}{d\theta^2} = -B\cos\theta$$

$$z = B\cos\theta + \frac{k}{h^2}$$

$$r = \frac{1}{k/h^2 + B\cos\theta} = \frac{h^2/k}{1+(Bh^2/k)\cos\theta} = \frac{h^2/k}{1+e\cos\theta}$$

Our task is to locate all points P such that the distance from F to P is a constant multiple of the distance from P to the directrix, the line *x* = *p*. In symbols, we want to locate all points P satisfying the relation *FP = e PD*





$$FB = r\cos\theta \qquad PD = FC - FB$$

$$PD = p - r\cos\theta \qquad FP = ePD$$

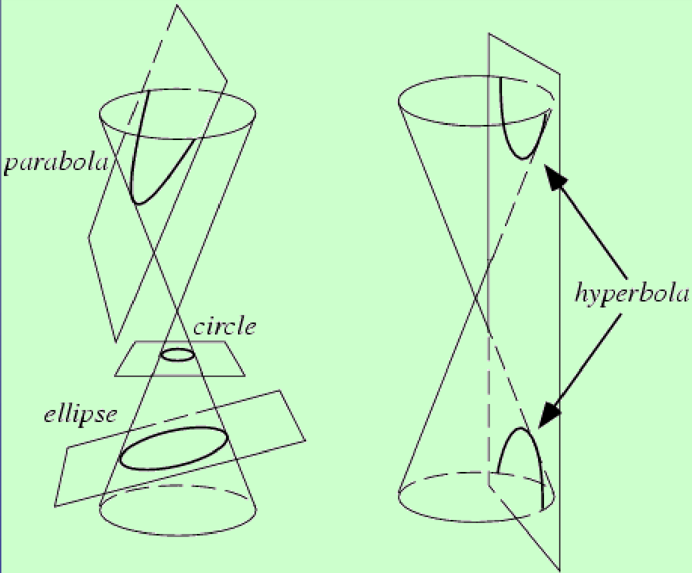
$$r = e(p - r\cos\theta) \qquad r = \frac{ep}{1+e\cos\theta}$$

Polar equation of the conic section

ORBIT CLASSIFICATIONS
Eccentricity

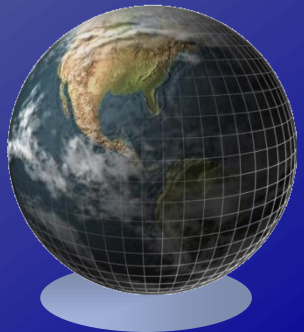
Circle | Ellipse | Parabola | Hyperbola

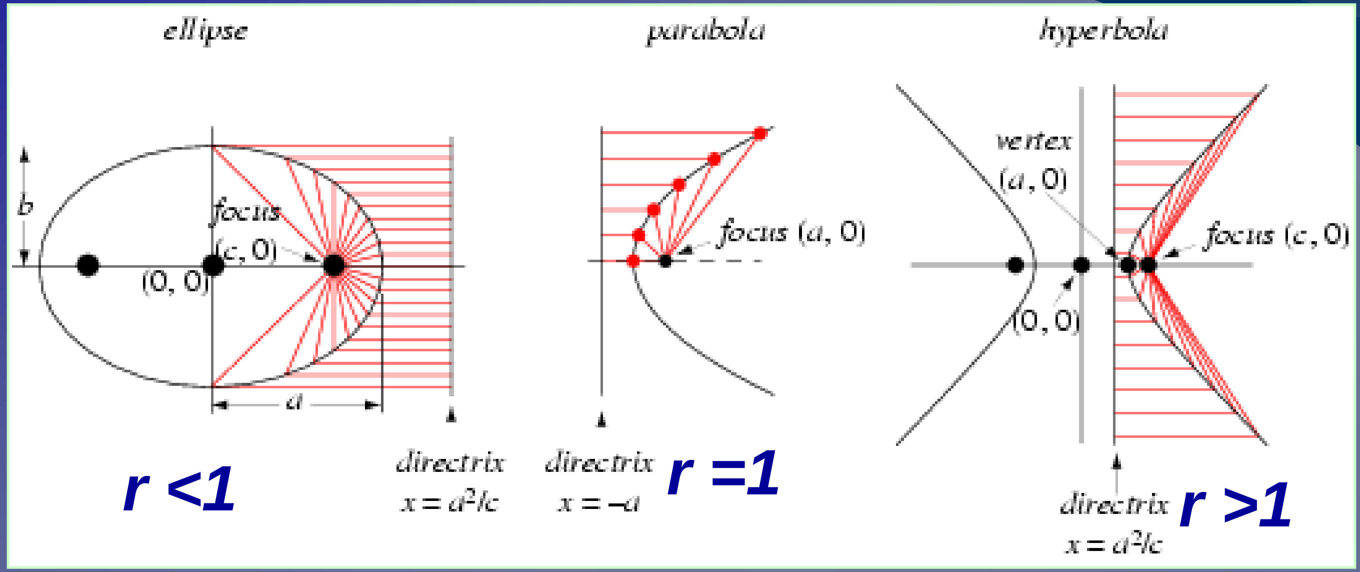
For a plane perpendicular to the axis of the cone, a circle is produced. For a plane that is not perpendicular to the axis and that intersects only a single nappe, the curve produced is either an ellipse or a parabola. The curve produced by a plane intersecting both nappes is a hyperbola.



$$r = \frac{h^2 / k}{1 + e\cos\theta}$$

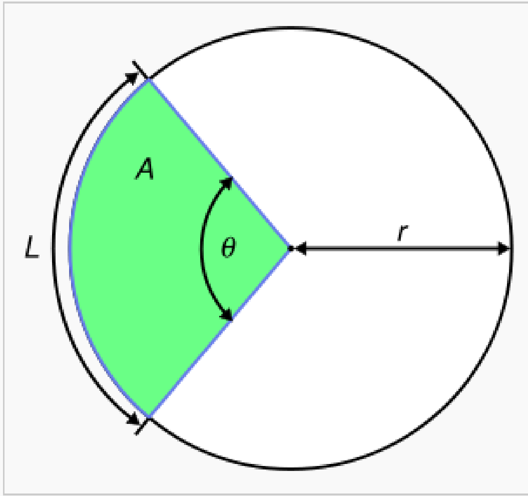
These remarks show that the orbit is a conic section with eccentricity $e = Bh^2/k$; and since the planets remain in the solar system and therefore move in closed orbits, we have Kepler's first law: the orbit of each planet is an ellipse with the sun at one focus.





The directrix of a conic section is the line which, together with the point known as the focus, serves to define a conic section as the locus of points whose distance from the focus is proportional to the horizontal distance from the directrix, with being the constant of proportionality. If the ratio *r =1*, the conic is a parabola, if *r <1*, it is an ellipse, and if *r >1*, it is a hyperbola.

A circular sector or *circle sector* also known as a *pie piece* is the portion of a circle enclosed by two radii and an arc. Its area can be calculated as described below.



A circular sector is shaded in green

Let θ be the central angle, in radians, and *r* the radius. The total area of a circle is πr^2. The area of the sector can be obtained by multiplying the circle's area by the ratio of the angle and 2π (because the area of the sector is proportional to the angle, and 2π is the angle for the whole circle):

$$A = \pi r^2 . \frac{\theta}{2\pi} = r^2 (\frac{\theta}{2}) = \frac{1}{2} r^2 \theta$$