Differential Equations

Lecture 13

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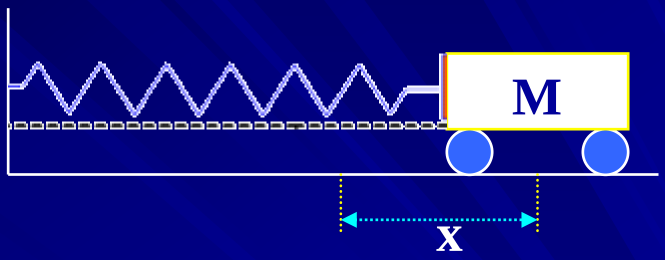
Applications of 2nd Order Linear

Vibrations in Mechanical Systems

How can we obtain the motion of the body, say, the displacement *x*(*t*) as function of time *t*? Now this motion is determined by Newton's second law



$$\vec{F} = M\vec{a} \qquad -kx = M\frac{d^2x}{dt^2} \qquad \frac{d^2x}{dt^2} + \frac{k}{M}x = 0$$

$$y'' + P(x)y' + Q(x)y = 0 \qquad P = 0,\ Q = k/M$$

$$x = e^{mt} \rightarrow x' = me^{mt} \rightarrow x'' = m^2e^{mt}$$

$$m^2e^{mt} + \frac{k}{M}e^{mt} = 0$$

$$m^2 + k/M = 0 \rightarrow m_1, m_2 = \pm\sqrt{-k/M}$$

$$x_1 = e^{i\sqrt{k/M}\,t}, \quad x_2 = e^{-i\sqrt{k/M}\,t}$$

$$\begin{cases} x_1 = \cos\sqrt{k/M}\,t + i\sin\sqrt{k/M}\,t \\ x_2 = \cos\sqrt{k/M}\,t - i\sin\sqrt{k/M}\,t \end{cases}$$

$$\begin{cases} x_3 = \frac{1}{2}x_1 + \frac{1}{2}x_2 = \frac{1}{2}(x_1 + x_2) = \cos\sqrt{k/M}t \\ x_4 = \frac{1}{2i}x_1 - \frac{1}{2i}x_2 = \frac{1}{2i}(x_1 - x_2) = \sin\sqrt{k/M}t \end{cases}$$

$$x = c_1 \cos\sqrt{k/M}\,t + c_2 \sin\sqrt{k/M}\,t$$

$$when\ \ t = 0, \quad x = x_0 \quad and \quad v = 0$$

$$x_0 = c_1$$

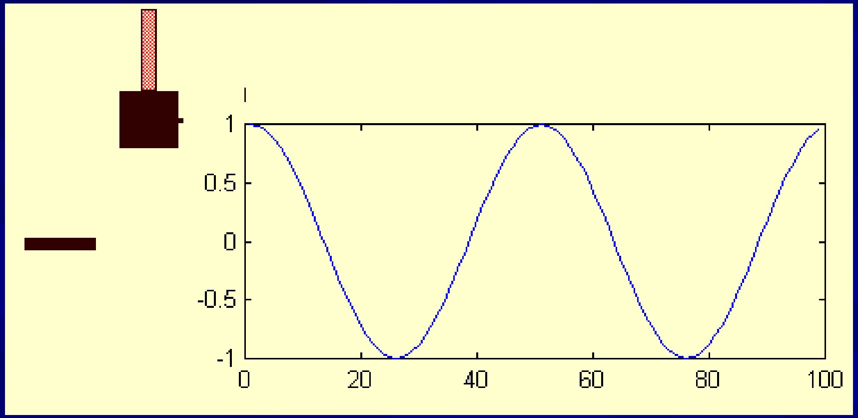
$$x = x_0 \cos\sqrt{k/M}\,t + c_2 \sin\sqrt{k/M}\,t$$

$$v = \frac{dx}{dt} = -x_0\sqrt{k/M}\,\sin\sqrt{k/M}\,t + c_2\sqrt{k/M}\,\cos\sqrt{k/M}\,t$$

$$0 = c_2\sqrt{k/M} \rightarrow c_2 = 0 \qquad x = x_0 \cos\sqrt{k/M}\,t$$

Simple harmonic vibration

Simple harmonic motion is the motion of a simple harmonic oscillator, a motion that is neither driven nor damped. The motion is periodic, as it repeats itself at standard intervals in a specific manner - described as being sinusoidal, with constant amplitude.



$$\cos\sqrt{k/M}T = \cos 2\pi$$

$$\sqrt{k/M}T = 2\pi \rightarrow T = 2\pi\sqrt{M/k}$$

Vibrations in Mechanical Systems

Damping vibrations: **If a frictional force (damping) proportional to the velocity is also present, the harmonic oscillator is described as a damped oscillator.**

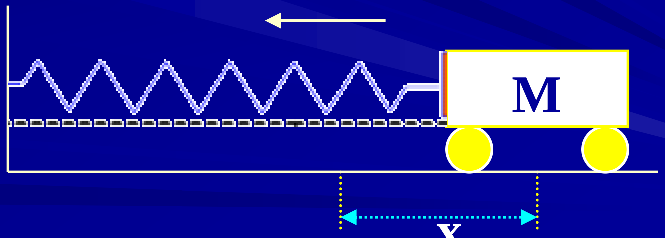
In such situation, the frequency of the oscillations is smaller than in the non-damped case, and the amplitude of the oscillations decreases with time. ***c* is *called the damping constant.***

$$\vec{F} = m\vec{a}$$

$$M\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} + \frac{c}{M}\frac{dx}{dt} + \frac{k}{M}x = 0$$

$$F_s = -kx$$



$$F_d = -c(dx/dt)$$

$$m^2 + \frac{c}{M}m + \frac{k}{M} = 0 \quad c/M = 2b, \qquad a = \sqrt{k/M}$$

$$m^2 + 2bm + a^2 = 0$$

$$m_1, m_2 = \frac{-2b \pm \sqrt{4b^2 - 4a^2}}{2} = -b \pm \sqrt{b^2 - a^2}$$

It is now most interesting that depending on the amount of damping (much, medium, or little) there will be three types of motion corresponding to the three cases.

$$\begin{cases} b^2 - a^2 > 0 \ \ or \ \ b > a \rightarrow distinct\ real\ roots\ m_1,\ m_2 \\ b^2 - a^2 = 0 \ \ or \ \ b = a \rightarrow equal\ real\ roots\ m \\ b^2 - a^2 < 0 \ \ or \ \ b < a \ \ distinct\ \text{complex}\ roots\ m_1,\ m_2 \end{cases}$$

Case 1) $b^2 - a^2 > 0$ Overdamping

If **the damping constant *c*** is so large that $c^2 > 4Mk$, then m_1 and m_2 are distinct real roots. In this case the corresponding general solution of equation is:

$$x = c_1 e^{m_1 t} + c_2 e^{m_2 t} \qquad v = \frac{dx}{dt} = c_1 m_1 e^{m_1 t} + c_2 m_2 e^{m_2 t}$$

$$t = 0, \quad x = x_0, \quad v = 0$$

$$x_0 = c_1 + c_2 \qquad 0 = c_1 m_1 + c_2 m_2 \qquad c_1 = -\frac{m_2}{m_1} c_2$$

$$x_0 = -\frac{m_2}{m_1} c_2 + c_2 \quad x_0 = \left(\frac{m_1 - m_2}{m_1}\right) c_2 \quad c_2 = \left(\frac{m_1}{m_1 - m_2}\right) x_0$$

$$c_1 = \frac{-m_2}{m_1 - m_2} x_0 \qquad c_2 = \left(\frac{m_1}{m_1 - m_2}\right) x_0$$

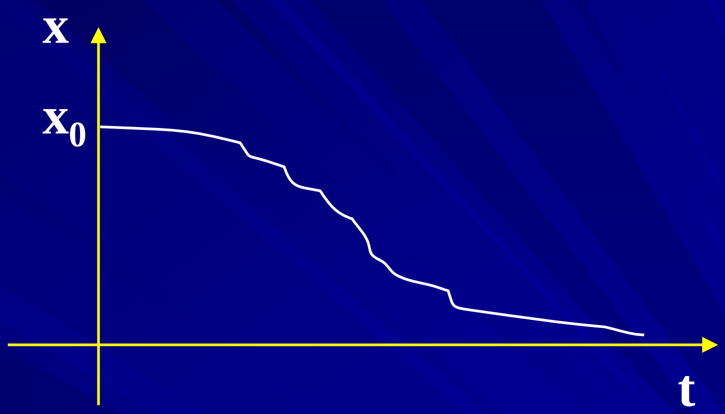
$$x = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

$$x = \frac{-m_2 x_0}{m_1 - m_2} e^{m_1 t} + \frac{m_1 x_0}{m_1 - m_2} e^{m_2 t}$$

$$x = \frac{x_0}{m_1 - m_2} (-m_2 e^{m_1 t} + m_1 e^{m_2 t})$$

$$x = \frac{x_0}{m_1 - m_2}\left(-m_2 e^{m_1 t} + m_1 e^{m_2 t}\right)$$

We see that in this case, damping takes out energy so quickly that the body does not oscillate.



overdamped

Case 2) $b^2=a^2$ Critical Damping

$$m_1, m_2 = \frac{-2b \pm \sqrt{4b^2 - 4a^2}}{2} = -b \pm \sqrt{b^2 - a^2} = -b = -a$$

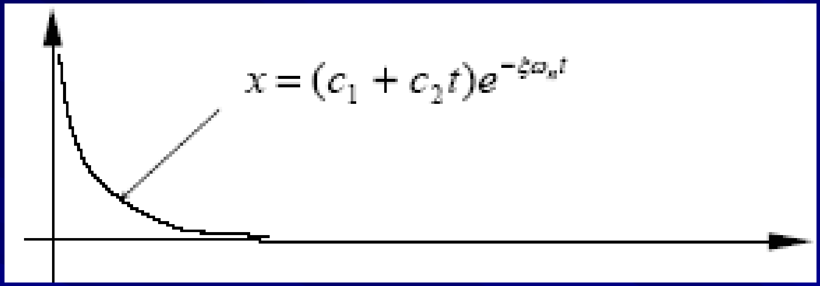
$$x = c_1 e^{-at} + c_2 t e^{-at} \qquad c_1 = x_0$$

$$v = \frac{dx}{dt} = -c_1 a e^{-at} + c_2 e^{-at} - a c_2 t e^{-at}$$

$$0 = -a c_1 + c_2 \qquad c_2 = a x_0$$

$$x = x_0 e^{-at} + x_0 a t e^{-at} = x_0 e^{-at}(1 + at)$$

Critical damping is the border case between nonoscillatory motions (Case I) and oscillations (Case III).



critically damped

Case 3) $b^2-a^2<0$ Underdamping

This is the most interesting case. It occurs if the damping constant *c* is so small that $c^2 < 4Mk$.

$$m_1, m_2 = \frac{-2b \pm \sqrt{4b^2 - 4a^2}}{2} = -b \pm i\sqrt{b^2 - a^2}$$

$$x_1 = e^{(-b+i\sqrt{a^2-b^2})t} \qquad x_2 = e^{(-b-i\sqrt{a^2-b^2})t}$$

$$\sqrt{a^2 - b^2} = \alpha$$

$$x_1 = e^{-bt}(\cos \alpha t + i \sin \alpha t)$$

$$x_2 = e^{-bt}(\cos \alpha t - i \sin \alpha t)$$

$$x_3 = \frac{1}{2}(x_1 + x_2) = e^{-bt}\cos\alpha t$$

$$x_4 = \frac{1}{2i}(x_1 - x_2) = e^{-bt}\sin\alpha t$$

$$x = e^{-bt}(c_1 \sin\alpha t + c_2 \cos\alpha t)$$

$$t = 0, \quad x = x_0, \quad v = 0 \rightarrow c_2 = x_0$$

$$v = \frac{dx}{dt} = -be^{-bt}(c_1 \sin\alpha t + c_2 \cos\alpha t) +$$

$$e^{-bt}(\alpha c_1 \cos\alpha t - \alpha c_2 \sin\alpha t)$$

$$0 = -bc_2 + \alpha c_1 \rightarrow bc_2 = \alpha c_1 \qquad c_1 = \frac{b}{\alpha}x_0$$

$$x = e^{-bt}\left(\frac{b}{\alpha}x_0 \sin \alpha t + x_0 \cos \alpha t\right) \qquad tg\theta = b/\alpha$$

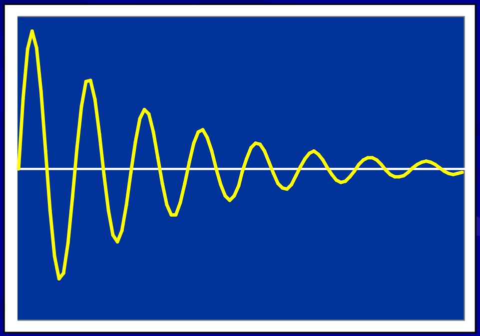
$$x = x_0 e^{-bt}(tg\theta \sin \alpha t + \cos \alpha t)$$

$$x = x_0 e^{-bt}\left(\frac{\sin \alpha t \sin \theta + \cos \alpha t \cos \theta}{\cos \theta}\right)$$

$$\frac{1}{\cos^2 \theta} = 1 + tg^2\theta = 1 + \frac{b^2}{\alpha^2} = \frac{\alpha^2 + b^2}{\alpha^2}$$

$$\frac{1}{\cos \theta} = \frac{\sqrt{\alpha^2 + b^2}}{\alpha}$$

$$x = x_0 \frac{\sqrt{\alpha^2 + b^2}}{\alpha} e^{-bt} \cos(\alpha t - \theta)$$



underdamped

$$\cos \alpha T = \cos 2\pi \qquad \alpha T = 2\pi \rightarrow T = \frac{2\pi}{\alpha}$$

$$b^2 - a^2 = -\alpha^2 \rightarrow \alpha = \sqrt{a^2 - b^2}$$

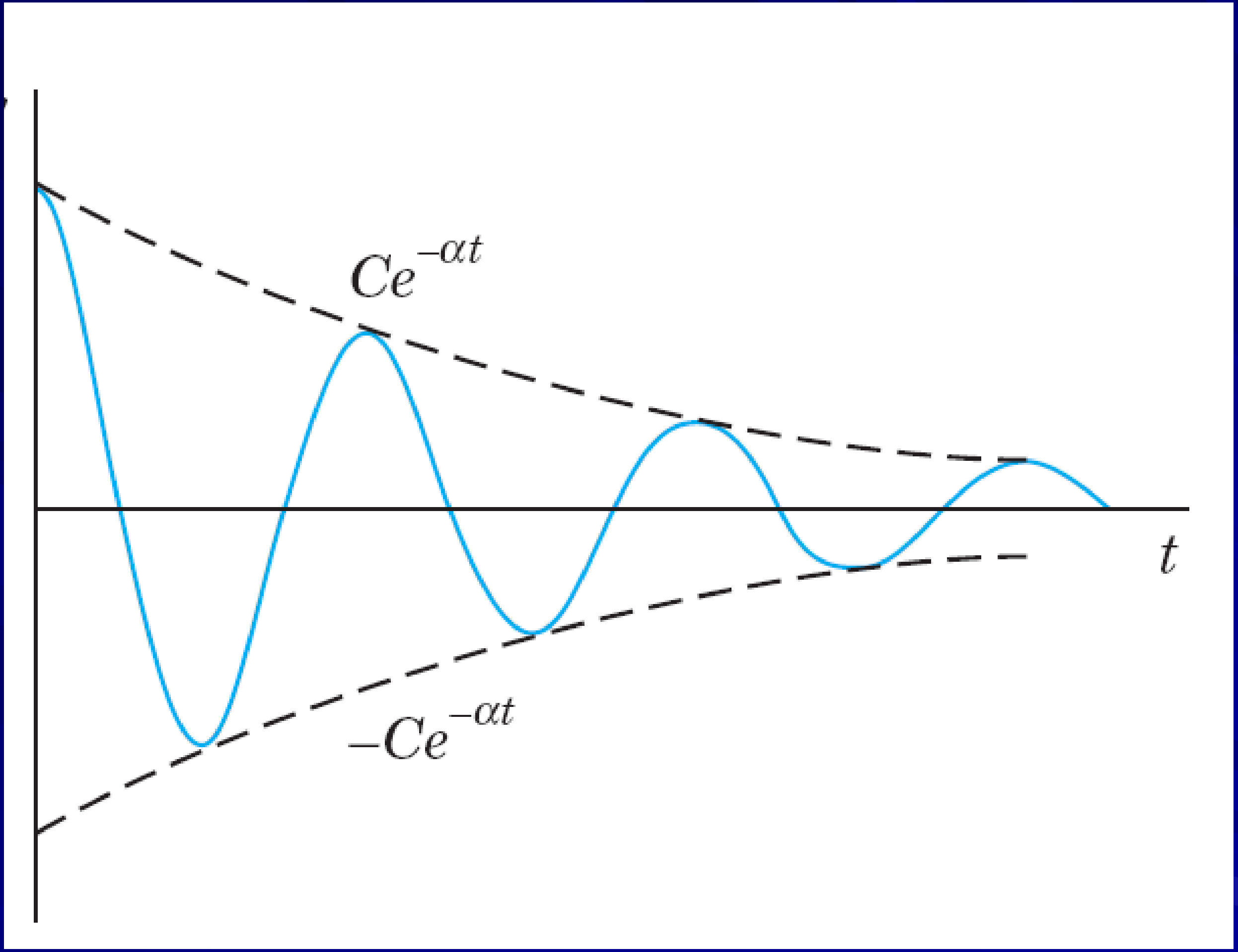
$$T = \frac{2\pi}{\sqrt{a^2 - b^2}} \qquad \frac{k}{M} = a^2, \frac{c}{M} = 2b$$

$$T = \frac{2\pi}{\sqrt{\frac{k}{M} - \frac{c^2}{4M^2}}} \qquad f = \frac{1}{2\pi}\sqrt{\frac{k}{M} - \frac{c^2}{4M^2}}$$

The natural frequency of the system

$$if \quad c = 0 \rightarrow f = \frac{1}{2\pi}\sqrt{k/M}$$

Damped oscillation in Case III



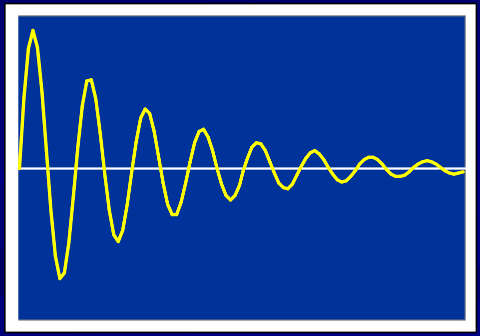
Damped Simple Harmonic Motion

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{M} - \frac{c^2}{4M^2}}$$

$$\omega = \sqrt{\omega_o^2 - (c/2M)^2}$$

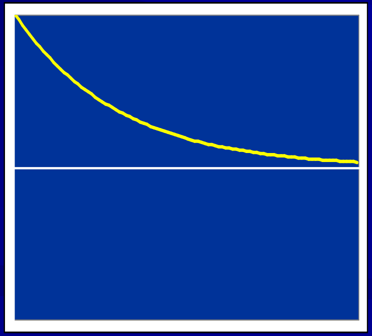
There are three mathematically distinct regimes

$\omega_o > c/2M$



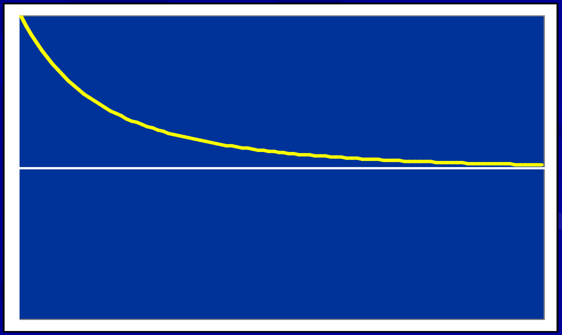
underdamped

$\omega_o = c/2M$



critically damped

$\omega_o < c/2M$



overdamped

Forced vibrations

The vibrations discussed earlier are called free vibrations because all the forces that affect the motion of the system are internal to the system. We extend our analysis to cover the case in which an external force acts on the mass.

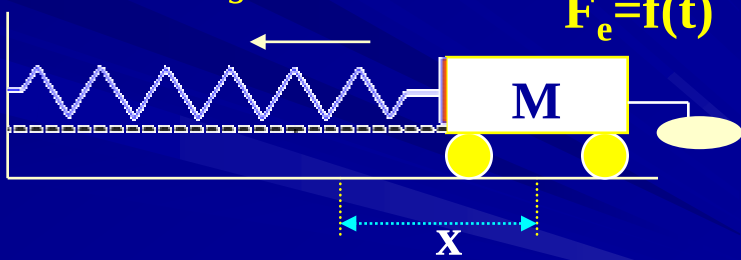
If an external time-dependent force is present, the harmonic oscillator is described as a driven oscillator.

$$F = Ma$$

$$M\frac{d^2x}{dt^2} = F_s + F_d + F_e$$

$$M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t)$$

$$F_s = -kx$$



$$F_d = -c(dx/dt)$$

$$M\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=F_0\cos\omega t$$

$$x(t)=x_p+x_g$$

$$x_p=A\sin\omega t+B\cos\omega t$$

$$x_p'=A\omega\cos\omega t-B\omega\sin\omega t$$

$$x_p''=-A\omega^2\sin\omega t-B\omega^2\cos\omega t$$

$$\sin\omega t\left[A(k-M\omega^2)-c\omega B\right]+\cos\omega t\left[B(k-M\omega^2)+c\omega A\right]$$
$$=F_0\cos\omega t$$

$$A(k-M\omega^2)-c\omega B=0 \qquad B(k-M\omega^2)+c\omega A=F_0$$

$$A=\frac{c\omega F_0}{(k-M\omega^2)^2+\omega^2c^2} \qquad B=\frac{(k-\omega^2M)F_0}{(k-\omega^2M)^2+\omega^2c^2}$$

$$x_p=\frac{c\omega F_0}{(k-M\omega^2)^2+\omega^2c^2}\sin\omega t+\frac{(k-\omega^2M)F_0}{(k-\omega^2M)^2+\omega^2c^2}\cos\omega t$$

$$x_p = \frac{F_0 c\omega}{(k - M\omega^2)^2 + \omega^2 c^2}\left[\sin\,\omega t + \frac{(k - \omega^2 M)}{c\omega}\cos\omega t\right]$$

$$tg\varphi = \frac{\omega c}{k - M\omega^2}$$

$$x_p = \frac{F_0 c\omega}{(k - M\omega^2)^2 + \omega^2 c^2}\left[\sin\,\omega t + \frac{\cos\varphi}{\sin\,\varphi}\cos\omega t\right]$$

$$x_p = \frac{F_0 c\omega}{\sin\,\varphi\left[(k - M\omega^2)^2 + \omega^2 c^2\right]}\cos(\omega t - \varphi)$$

$$1 + tg^2\varphi = \frac{1}{\cos^2\varphi}$$

$$tg\,\varphi = \frac{\omega c}{k - m\omega^2}$$

$$\sin^2\varphi = 1 - \cos^2\varphi = 1 - \frac{(k - m\omega^2)^2}{(k - m\omega^2)^2 + c^2\omega^2}$$

$$\sin\varphi = \frac{c\omega}{\sqrt{(k - m\omega^2)^2 + \omega^2 c^2}}$$

$$x = e^{-bt}(c_1 \cos \alpha t + c_2 \sin \alpha t) + \frac{F_0}{\sqrt{(k - M\omega^2)^2 + \omega^2 c^2}} \cos(\omega t - \varphi)$$

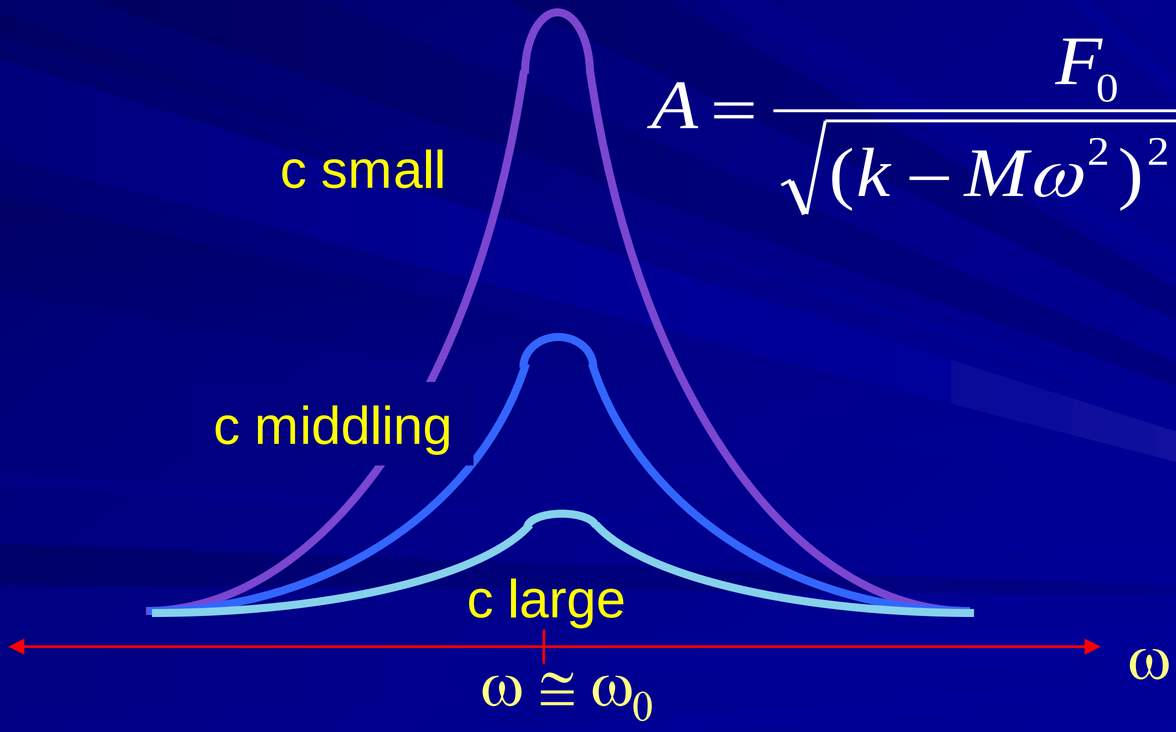
The solution to the driven harmonic oscillator has a transient and a steady-state part. The transient solution is the solution to the homogeneous differential equation of motion which has been combined with the particular solution and forced to fit the physical boundary conditions of the problem at hand. The form of this transient solution is that of the undriven damped oscillator and as such can be underdamped, overdamped, or critically damped.

Driven SHM with Resistance

- Apply a sinusoidal force, $F_0 \cos(\omega t)$, and now consider what A and c do,

$$\frac{d^2x}{dt^2} + \frac{b}{M}\frac{dx}{dt} + \frac{k}{M}x = \frac{F}{M}\cos\omega t$$

$$A = \frac{F_0}{\sqrt{(k - M\omega^2)^2 + c^2\omega^2}}$$



Forced Vibrations: Resonance

$$\omega_0 = \sqrt{k/m} = \omega$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{M} - \frac{c^2}{4M^2}}$$

The natural frequency of the system