Differential Equations

Lecture 12

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The Method of Undetermined Coefficients

$$y'' + P(x)y' + Q(x)y = R(x)$$

$$y'' + P(x)y' + Q(x)y = 0$$

$$y(x) = y_g(x) + y_p(x) \text{ g.s.}$$

In this section we use the method of undetermined coefficients to find a particular solution $y_p(x)$ to the nonhomogeneous equation.

The method of undetermined coefficients is usually limited to when $P(x)$ and $Q(x)$ are constant, and $R(x)$ is a exponential, sine or cosine or polynomial, function.

Example: Find P.S. $y'' + 4y = 3\cos 2x$

$$y_p = A\sin 2x + B\cos 2x$$

$$y_p' = 2A\cos 2x - 2B\sin 2x, \; y'' = -4A\sin 2x - 4B\cos 2x$$

$$(-4A\sin 2x - 4B\cos 2x) + 4(A\sin 2x + B\cos 2x) = 3\cos 2x$$

$$(-4A + 4A)\sin 2x + (-4B + 4B)\cos 2x = 3\cos 2x$$

$0 = 3\cos 2x$ **Thus no particular solution exists of the form y_p**

$$y'' + 4y = 0 \;\; \Rightarrow \;\; y(t) = c_1\cos 2x + c_2\sin 2x$$

Thus our assumed particular solution solves homogeneous equation instead of the nonhomogeneous equation.

$$y_p = Ax\sin 2x + Bx\cos 2x$$

$$y_p' = A\sin 2x + 2Ax\cos 2x + B\cos 2x - 2Bx\sin 2x$$

$$y_p'' = 2A\cos 2x + 2A\cos 2x - 4Ax\sin 2x - 2B\sin 2x - 2B\sin 2x - 4Bx\cos 2x$$

$$= 4A\cos 2x - 4B\sin 2x - 4Ax\sin 2x - 4Bx\cos 2x$$

$$y'' + 4y = 3\cos 2x$$

$$4A\cos 2x - 4B\sin 2x = 3\cos 2x$$

$$A = 3/4, \qquad B = 0$$

$$y_p = \frac{3}{4}x\sin 2x$$

$$y_g = c_1\cos 2x + c_2\sin 2x + \frac{3}{4}x\sin 2x$$

Case 3: $y'' + py' + qy = a_0 + a_1 x + \ldots + a_n x^n$

$$y_p = A_0 + A_1 x + \ldots + A_n x^n$$

$$y_p = x(A_0 + A_1 x + \ldots + A_n x^n)$$

The above discussions show that the form of a particular solution of equation $y'' + P(x)y' + Q(x)y = R(x)$ **can often be inferred from the form of the right-hand member *R(x)*.**

Example: Find P.S.

$$y'' - 3y' - 4y = 4x^2 - 1$$

$$y_p = A_2 x^2 + A_1 x + A_0$$

$$y_p' = 2A_2 x + A_1, \qquad y_p'' = 2A_2$$

$$2A_2 - 3(2A_2 x + A_1) - 4(A_2 x^2 + A_1 x + A_0) = 4x^2 - 1$$

$$-4A_2 x^2 - (6A_2 + 4A_1)x + (2A_2 - 3A_1 - 4A_0) = 4x^2 - 1$$

$$-4A_2 = 4, \qquad 6A_2 + 4A_1 = 0, \qquad 2A_2 - 3A_1 - 4A_0 = -1$$

$$A_2 = -1, \qquad A_1 = 3/2, \qquad A_0 = -11/8$$

$$y_p = -x^2 + \frac{3}{2}x - \frac{11}{8}$$

Example: Find P.S. $y'' + 4y = 8x^2$

$$y_p = A_2x^2 + A_1x + A_0$$

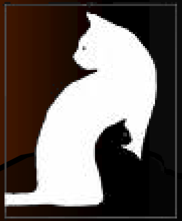
$$y_p' = 2A_2x + A_1 \qquad y_p'' = 2A_2$$

$$2A_2 + 4(A_2x^2 + A_1x + A_0) = 8x^2$$

$$4A_2x^2 + 4A_1x + (2A_2 + 4A_0) = 8x^2$$

$$A_2 = 2, \qquad A_1 = 0, \qquad A_0 = -1$$

$$y_p = 2x^2 - 1$$



$$y'' + P(x)y' + Q(x)y = R(x)$$

Particular case 1: If Q=0, polynomial is n+1 degree.

$$y'' + 5y' = 3 + x \qquad y_p = A_2x^2 + A_1x + A_0$$

Particular case 2: If P=Q=0, polynomial is n+2 degree.

$$y'' = 1 + x \qquad y_p = A_3x^3 + A_2x^2 + A_1x + A_0$$

Example: Find P.S. $y'' - 3y' - 4y = -8e^{x}\cos 2x$

$$y_p = Ae^{x}\cos 2x + Be^{x}\sin 2x$$

$$y_p' = Ae^{x}\cos 2x - 2Ae^{x}\sin 2x + Be^{x}\sin 2x + 2Be^{x}\cos 2x$$

$$= (A + 2B)e^{x}\cos 2x + (-2A + B)e^{x}\sin 2x$$

$$y_p'' = (A + 2B)e^{x}\cos 2x - 2(A + 2B)e^{x}\sin 2x +$$

$$(-2A + B)e^{x}\sin 2x + 2(-2A + B)e^{x}\cos 2x$$

$$= (-3A + 4B)e^{x}\cos 2x + (-4A - 3B)e^{x}\sin 2x$$

$$A = \frac{10}{13},\ B = \frac{2}{13} \quad \Rightarrow y_p = \frac{10}{13}e^{x}\cos 2x + \frac{2}{13}e^{x}\sin 2x$$

$$If\ \ R(x) = R_1(x) + R_2(x) + \ldots \rightarrow y_p = y_{1p} + y_{2p} + \ldots$$

$$y'' - 3y' - 4y = 3e^{2x} + 2\sin x - 8e^x \cos 2x$$

$$y'' - 3y' - 4y = 3e^{2x}$$

$$y'' - 3y' - 4y = 2\sin x$$

$$y'' - 3y' - 4y = -8e^x \cos 2x$$

$$y = -\frac{1}{2}e^{2x} + \frac{3}{17}\cos x - \frac{5}{17}\sin x + \frac{10}{13}e^x \cos 2x + \frac{2}{13}e^x \sin 2x$$

Problem Page No 97- 1 – c- Find g.s.

$$y'' + 10y' + 25y = 14e^{-5x}$$

$$y'' + 10y' + 25y = 0 \qquad y = e^{mx}$$

$$m^2 + 10m + 25 = 0 \qquad m_1 = m_2 = -5$$

$$y_g = c_1e^{-5x} + c_2xe^{-5x} \qquad y_p = Ae^{-5x}$$

$$y_p = Ax^2e^{-5x} \qquad y_p' = 2Axe^{-5x} - 5Ax^2e^{-5x}$$

$$y_p'' = Ae^{-5x}(25x^2 - 20x + 2)$$

$$y'' + 10y' + 25y = 14e^{-5x}$$

$$A = 7 \qquad y_p = Ax^2e^{-5x} \qquad y_p = 7x^2e^{-5x}$$

$$y = y_g + y_p = c_1e^{-5x} + c_2xe^{-5x} + 7x^2e^{-5x}$$

$R(x)$	y_p
e^{ax}	Ae^{ax}
$\sin bx$	$A\sin bx + B\cos bx$
$\cos bx$	$A\sin bx + B\cos bx$
$ax^n \ (n = 0,1,\ldots)$	$A_0 + A_1 x + \ldots + A_n x^n$
$e^{ax}\sin bx$	$e^{ax}(A\sin bx + B\cos bx)$
$e^{ax}\cos bx$	$e^{ax}(A\sin bx + B\cos bx)$

The Method of Variation of Parameters

In this section we will learn the variation of parameters method to solve the nonhomogeneous equation.

$$y'' + p(x)y' + Q(x)y = R(x) \quad (1)$$

$$y'' + p(x)y' + Q(x)y = 0 \quad (2)$$

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \quad \text{g.s.of } (2)$$

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

$$y' = (v_1 y_1' + v_2 y_2') + (v_1' y_1 + v_2' y_2)$$

$$(v_1' y_1 + v_2' y_2) = 0$$

$$y' = v_1 y_1' + v_2 y_2'$$

$$y'' = v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2'$$

$$v_1(y_1'' + P y_1' + Q y_1) + v_2(y_2'' + P y_2' + Q y_2) + v_1' y_1' + v_2' y_2' = R(x)$$

$$v_1' y_1' + v_2' y_2' = R(x)$$

$$\begin{cases} v_1' y_1 + v_2' y_2 = 0 \\ v_1' y_1' + v_2' y_2' = R(x) \end{cases} \qquad \begin{cases} y_1' v_1' y_1 + y_1' v_2' y_2 = 0 \\ y_1 v_1' y_1' + y_1 v_2' y_2' = y_1 R(x) \end{cases}$$

$$v_2'(y_1 y_2' - y_1' y_2) = y_1 R(x)$$

$$v_1'(y_1 y_2' - y_1' y_2) = -y_2 R(x)$$

$$v_1' = \frac{-y_2 R(x)}{W(y_1, y_2)} \qquad v_2' = \frac{y_1 R(x)}{W(y_1, y_2)}$$

$$v_1 = \int \frac{-y_2 R(x)}{W(y_1, y_2)} dx \qquad v_2 = \int \frac{y_1 R(x)}{W(y_1, y_2)} dx$$

$$y_p(x) = v_1(x) y_1(x) + v_2(x) y_2(x)$$

$$y_p(x) = y_1 \int \frac{-y_2 R(x)}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 R(x)}{W(y_1, y_2)} dx$$

Example: Find P.S. $y'' + y = \csc x$

$$y'' + y = 0 \qquad m^2 + 1 = 0 \rightarrow m = \pm i$$

$$y_1 = e^{m_1 x} = e^{ix} \qquad y_2 = e^{m_2 x} = e^{-ix}$$

$$y_g(x) = c_1 \sin x + c_2 \cos x$$

$$W(y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1$$

$$v_1 = \int \frac{-y_2 R(x)}{W(y_1, y_2)} dx = \int \frac{-\cos x \csc x}{-1} dx = \int \frac{\cos x}{\sin x} dx$$

$$= \ln(\sin x)$$

$$v_2 = \int \frac{y_1 R(x)}{W(y_1, y_2)} dx = \int \frac{\sin x \csc x}{-1} dx = -x$$

$$y_p(x) = v_1(x) y_1(x) + v_2(x) y_2(x)$$

$$y_p(x) = \ln(\sin x) \sin x - x \cos x$$

Example: Find G.S. $y'' + 2y' + 2y = \dfrac{e^{-x}}{\cos^3 x}$

$y'' + 2y' + 2y = 0 \qquad m^2 + 2m + 2 = 0$

$$m_1, m_2 = \frac{-p \pm \sqrt{p^2 - 4q}}{2} = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$$y_g(x) = e^{-x}(c_1 \cos x + c_2 \sin x)$$

$$W(y_1, y_2) = \begin{vmatrix} e^{-x}\cos x & e^{-x}\sin x \\ -e^{-x}\cos x - e^{-x}\sin x & -e^{-x}\sin x + e^{-x}\cos x \end{vmatrix} =$$

$$= e^{-2x}$$

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

$$y_p(x) = y_1 \int \frac{-y_2 R(x)}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 R(x)}{W(y_1, y_2)} dx$$

$$y_p(x) = -e^{-x} \cos x \int \frac{e^{-2x} \sin x}{e^{-2x} \cos^3 x} dx +$$

$$e^{-x} \sin x \int \frac{e^{-2x} \cos x}{e^{-2x} \cos^3 x} dx =$$

$$= -e^{-x} \cos x \int \frac{\sin x}{\cos^3 x} dx + e^{-x} \sin x \int \sec^2 x dx$$

$$= \frac{1}{2} e^{-x} \cos x \cos^{-2} x + e^{-x} \sin x tgx$$

$$y(x) = e^{-x}(c_1 \cos x + c_2 \sin x) + y_p(x)$$

Example: Find G.S. $y'' + 2y' + y = 4e^{-x} \ln x$

$$y'' + 2y' + y = 0 \qquad m^2 + 2m + 1 = 0$$

$$m_1 = m_2 = -1 \qquad y_g(x) = (c_1 + c_2 x)e^{-x}$$

$$W(y_1, y_2) = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix} = e^{-2x}$$

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

$$y_p(x) = y_1 \int \frac{-y_2 R(x)}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 R(x)}{W(y_1, y_2)} dx$$

$$y_p(x) = -e^{-x}\int \frac{4e^{-2x} x \ln x}{e^{-2x}}dx + xe^{-x}\int \frac{4e^{-2x}\ln x}{e^{-2x}}dx$$

$$y_p(x) = -4e^{-x}\left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right) + 4xe^{-x}(x\ln x - x)$$

$$= x^2 e^{-x}(2\ln x - 4)$$

$$y(x) = e^{-x}(c_1 + c_2 x) + x^2 e^{-x}(2\ln x - 4)$$

Problem Page No 101-3-a Find p.s.

$$y'' + 4y = tg2x$$

$$m^2 + 4 = 0 \rightarrow m = \pm\sqrt{-4} = \pm 2i$$

$$y_1 = e^{i2x}, \ y_2 = e^{-i2x}$$

$$y_3 = \cos 2x, \quad y_4 = \sin 2x$$

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

$$y_p(x) = y_1 \int \frac{-y_2 R(x)}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 R(x)}{W(y_1, y_2)} dx$$

$$W(y_1, y_2) = \begin{vmatrix} \cos 2x & \text{sin2x} \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2(\cos^2 2x + \sin^2 2x) = 2$$

$$y_p(x) = \cos 2x \int \frac{-\sin 2x \tan 2x}{2} dx +$$

$$\sin 2x \int \frac{\cos 2x \tan 2x}{2} dx$$

$$= -\frac{1}{2} \cos 2x \int \sin 2x \tan 2x dx + \frac{1}{2} \sin 2x \int \sin 2x dx$$

$$y_p(x) = -\frac{1}{2}\cos 2x \int \sin 2x \tan 2x dx + \frac{1}{2}\sin 2x(-\frac{1}{2}\cos 2x)$$

$$u = \tan 2x \quad \to du = \frac{1}{\cos^2 2x} dx$$

$$dv = \sin 2x dx \quad \to v = -\frac{1}{2}\cos 2x$$

$$\int u dv = uv - \int v du \quad = -\frac{1}{2}\sin 2x + \frac{1}{2}\int \frac{dx}{\cos 2x}$$

$$= -\frac{1}{2}\sin 2x + \frac{1}{2}\ln(\sec 2x + \tan 2x)$$

$$y_p(x) = -\frac{1}{4}\cos 2x \ln(\sec 2x + \tan 2x)$$

Problem Page No 101- 3 – d- Find p.s.

$$y'' + 2y' + 5y = e^{-x}\sec 2x$$

$$y'' + 2y' + 5y = 0$$

$$m_{1,2} = -1 \pm 2i$$

$$y_1 = e^{-x}\cos 2x, \quad y_2 = e^{-x}\sin 2x$$

$$y_p(x) = y_1 \int \frac{-y_2 R(x)}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 R(x)}{W(y_1, y_2)} dx$$

$$W(y_1, y_2) = \begin{vmatrix} e^{-x}\cos 2x & \mathrm{e}^{-x}\mathrm{sin2x} \\ -e^{x}(\cos 2x + 2\sin 2x & e^{-x}(2\cos 2x - \sin 2x \end{vmatrix} = 2e^{-2x}$$

$$y_p(x) = -\frac{1}{2}e^{-x}\cos 2x\int \tan 2x dx + \frac{1}{2}e^{-x}\sin 2x\int dx$$

$$y_p(x) = -\frac{1}{2}e^{-x}\cos 2x[\ \ (-\frac{1}{2}\ln(\cos 2x)\ \ ]$$

$$+\frac{1}{2}e^{-x}\sin 2x(x)$$

$$y_p(x) = \frac{1}{4}e^{-x}\cos 2x[\ \ (\ln(\cos 2x)\ \ ]$$

$$+\frac{1}{2}e^{-x}x\sin 2x$$

Problem Page No 101- 4 - a Find g.s.

$$(x^2-1)y''-2xy'+2y=(x^2-1)^2$$

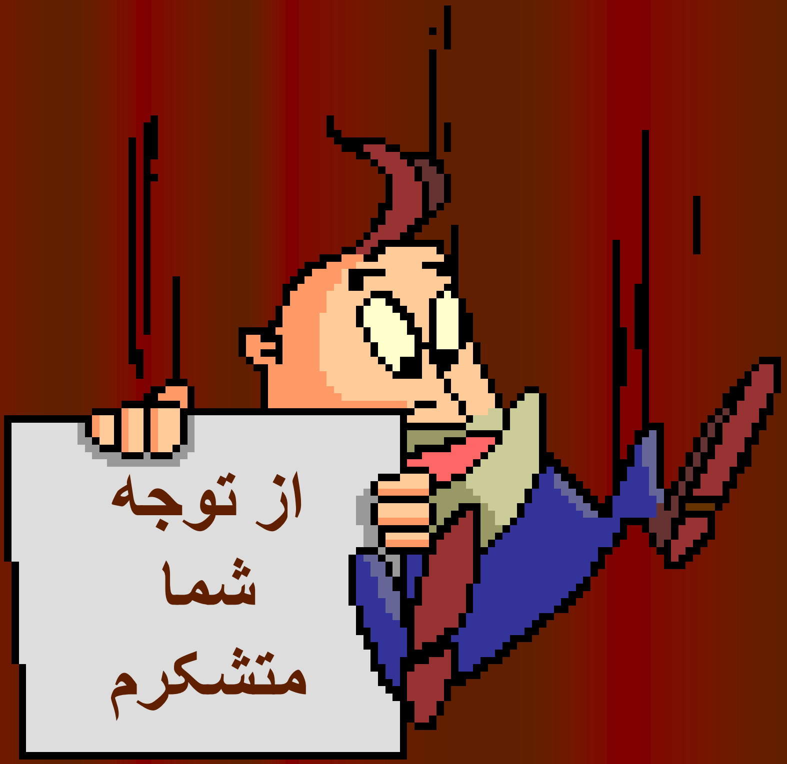
$$y''-\frac{2x}{(x^2-1)}y'+\frac{2}{(x^2-1)}y=(x^2-1)$$

$$y_1=e^x \text{ or } x \text{ or } x^2 \qquad y_2=vy_1 \qquad v=\int\frac{1}{y_1^2}e^{-\int Pdx}dx$$

$$y_2=vy_1=x\int\frac{1}{x^2}e^{-\int\frac{2x}{1-x^2}dx}dx=\int\frac{1}{x^2}e^{\ln(1-x^2)}dx$$

$$y_g=c_1y_1+c_2y_2$$

$$y_p(x)=y_1\int\frac{-y_2R(x)}{W(y_1,y_2)}dx+y_2\int\frac{y_1R(x)}{W(y_1,y_2)}dx$$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