Differential Equations

Lecture 11

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Homogeneous Equations with Constant Coefficients

$$y'' + py' + qy = 0$$

whose coefficients *p* and *q* are constant.

$$y = e^{mx} \qquad y' = me^{mx} \qquad y'' = m^2 e^{mx}$$

$$m^2 e^{mx} + pme^{mx} + qe^{mx} = 0$$

$$(m^2 + pm + q)e^{mx} = 0$$

Characteristic Equation

$$m^2 + pm + q = 0$$

$$m_1, m_2 = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

Case 1: **Distinct real roots m_1, m_2**

$$m_1, m_2 = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

$$p^2 - 4q > 0 \quad y_g(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} \quad \frac{e^{m_1 x}}{e^{m_2 x}} \neq k$$

Case 2: **Equal real roots $m_1 = m_2 = m$**

$$p^2 - 4q = 0 \qquad y_1 = e^{mx} = e^{(-p/2)x}$$

$$v = \int \frac{1}{y_1^2} e^{-\int P dx} dx = \int \frac{1}{e^{-px}} e^{-px} dx = x$$

$$y_2 = v y_1 \qquad y_g(x) = c_1 e^{mx} + c_2 x e^{mx}$$

Case 3: **Distinct complex roots m_1, m_2**

$$m_1, m_2 = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

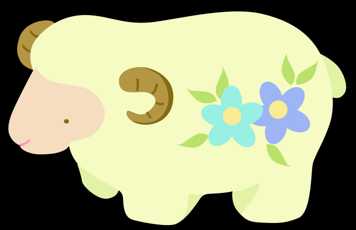
$$p^2 - 4q < 0$$

$$m_1, m_2 = a \pm bi \qquad i^2 = -1 \qquad e^{i\theta} = \cos\theta + i\sin\theta$$

$$y_1 = e^{m_1 x} = e^{(a+ib)x} \qquad y_1 = e^{ax}(\cos bx + i\sin bx)$$

$$y_2 = e^{m_2 x} = e^{(a-ib)x} \qquad y_2 = e^{ax}(\cos bx - i\sin bx)$$

$$\frac{y_1}{y_2} = \frac{e^{(a+ib)x}}{e^{(a-ib)x}} = e^{2ibx} \neq k$$



$$y_3 = \frac{1}{2}(y_1 + y_2) = \mathrm{e}^{ax}\cos bx \qquad c_1 = c_2 = \frac{1}{2}$$

$$y_4 = \frac{1}{2i}(y_1 - y_2) = \mathrm{e}^{ax}\sin bx \qquad c_1 = c_2 = \frac{1}{2i}$$

$$\frac{y_3}{y_4} = \cot bx \neq k$$

$$y_g(x) = c_1 y_3 + c_2 y_4 = e^{ax}(c_1 \cos bx + c_2 \sin bx)$$

Sec way

$$\frac{y_1}{y_2}=\frac{e^{(a+ib)x}}{e^{(a-ib)x}}=e^{2ibx}\neq k$$

$$y_g(x)=c_1e^{(a+ib)x}+c_2e^{(a-ib)x}$$

$$c_1=\frac{1}{2}(A-iB)\quad,\quad c_2=\frac{1}{2}(A+iB)$$

$$y_g(x)=e^{ax}\left(\frac{A-iB}{2}e^{ibx}+\frac{A+iB}{2}e^{-ibx}\right)$$

$$y_g(x) = e^{ax}\Big[\ \frac{1}{2}(A - iB)(\cos bx + i\sin bx)$$

$$+\frac{1}{2}(A + iB)(\cos bx - i\sin bx)\ \Big]$$

$$y_g(x) = e^{ax}(A\cos bx + B\sin bx)$$

Summary of Cases I–III

Case	Root of $m_1, m_2 = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$	**Basis of solution**	**General solution**
1	**Distinct real** ***m_1, m_2***	$e^{m_1 x}, e^{m_2 x}$	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
2	**Equal real roots** ***$m = -p/2$***	$e^{-px/2}, xe^{-px/2}$	$y = (c_1 + c_2 x) e^{-px/2}$
3	**Distinct complex roots** ***$m_1 = -p/2 + ib$*** ***$m_2 = -p/2 - ib$***	$e^{-px/2} \sin bx$ $e^{-px/2} \cos bx$	$y = e^{-px/2}(c_1 \cos bx$ $+ c_2 \sin bx)$

Example: Find G.S. $y'' - 8y' + 15y = 0$

$$y = e^{mx} \qquad y' = me^{mx} \qquad y'' = m^2 e^{mx}$$

$$m^2 - 8m + 15 = 0$$

$$m_1, m_2 = \frac{-p \pm \sqrt{p^2 - 4q}}{2} = \frac{8 \pm \sqrt{64 - 60}}{2} = 5 \,\&\, 3$$

$$y(x) = c_1 e^{5x} + c_2 e^{3x}$$

Example: Find G.S. $4y'' - 12y' + 9y = 0$

$$y'' - 3y' + \frac{9}{4}y = 0$$

$$y = e^{mx} \qquad y' = me^{mx} \qquad y'' = m^2 e^{mx}$$

$$m^2 - 3m + 9/4 = 0$$

$$m_1, m_2 = \frac{-p \pm \sqrt{p^2 - 4q}}{2} = \frac{3 \pm \sqrt{9-9}}{2} = m_1 = m_2 = 3/2$$

$$y(x) = c_1 e^{\frac{3}{2}x} + c_2 x e^{\frac{3}{2}x}$$

Example: Find G.S. $y'' - 4y' + 13y = 0$

$$m^2 - 4m + 13 = 0$$

$$m_1, m_2 = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i$$

$$y(x) = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$$

Example: Find P.S.

$$y'' + y' - 2y = 0, \quad y(0) = 4, \quad y'(0) = -5$$

$$m^2 + m - 2 = 0, \qquad y_g = c_1 e^{x} + c_2 e^{-2x}$$

$$\Rightarrow m = 1, -2 \qquad y'_g = c_1 e^{x} - 2c_2 e^{-2x}$$

$$y_g(0) = c_1 + c_2 = 4$$

$$y'_g(0) = c_1 - 2c_2 = -5$$

$$\rightarrow c_1 = 1, \quad c_2 = 3$$

$$y_p = e^{x} + 3e^{-2x}$$

The Method of Undetermined Coefficients

$$y'' + P(x)y' + Q(x)y = R(x)$$

$$y'' + P(x)y' + Q(x)y = 0$$

$$y(x) = y_g(x) + y_p(x) \quad \text{g.s.}$$

In this section we use the method of undetermined coefficients to find a particular solution $y_p(x)$ to the nonhomogeneous equation.

The method of undetermined coefficients is usually limited to when $P(x)$ and $Q(x)$ are constant, and $R(x)$ is a exponential, sine or cosine or polynomial, function.

$$R(x) = \begin{cases} e^{ax} \\ \sin bx, \cos bx \\ a_0 + a_1 x + \ldots + a_n x^n \end{cases}$$

Case 1: $y'' + py' + qy = e^{ax}$ *

$$y_p = Ae^{ax} \Rightarrow y_p' = aAe^{ax}, \quad y_p'' = a^2 Ae^{ax}$$

$$A(a^2 + pa + q)e^{ax} = e^{ax}$$

$$A = \frac{1}{a^2 + pa + q} \qquad m^2 + pm + q = 0$$

$$y_p = Axe^{ax} \rightarrow y_p' = Ae^{ax} + aAxe^{ax}$$

$$y_p'' = aAe^{ax} + aAe^{ax} + a^2 Axe^{ax}$$

$$y'' + py' + qy = e^{ax}$$

$$A\underbrace{(a^2 + pa + q)}_{0}xe^{ax} + A(2a + p)e^{ax} = e^{ax}$$

$$A = \frac{1}{2a + p}$$

This gives a valid coefficient for above solution except *a=-p/2*

$$y_p = Ax^2e^{ax}$$

$$A(a^2 + pa + q)x^2e^{ax} + 2A(2a + p)xe^{ax} + 2Ae^{ax} = e^{ax}$$

$$A = \frac{1}{2}$$

To summarize: if *a* is not a root of the auxiliary equation *$m^2+pm+q=0$* then * has a P.S. of the form *Ae^{ax}*.

If *a* is a simple root of *$m^2+pm+q=0$ then* has no solution of the form Ae^{ax}* but does have one of the form *Axe^{ax}*.

If *a* is double root, then * has no solution of the form *Axe^{ax}* but does have one of the form *Ax^2e^{ax}*

$$y'' + py' + qy = e^{ax} \quad *$$

Example: find G.S. $y'' - 3y' - 4y = 3e^{2x}$

$$y'' - 3y' - 4y = 0 \qquad m^2 - 3m - 4 = 0$$

$$m_1, m_2 = \frac{3 \pm \sqrt{9+16}}{2} = 4, -1 \qquad y_g = c_1 e^{4x} + c_2 e^{-x}$$

$$a = 2, \quad m \neq a$$

$$y_p = Ae^{2x} \Rightarrow y_p' = 2Ae^{2x}, \; y_p'' = 4Ae^{2x}$$

$$4Ae^{2x} - 6Ae^{2x} - 4Ae^{2x} = 3e^{2x}$$

$$-6Ae^{2x} = 3e^{2x} \qquad A = -1/2 \qquad y_p = -\frac{1}{2} e^{2x}$$

$$y(x) = c_1 e^{4x} + c_2 e^{-x} - \frac{1}{2} e^{2x}$$

Example: find G.S. $y'' - 3y' - 4y = 3e^{-x}$

$$y'' - 3y' - 4y = 0 \qquad m^2 - 3m - 4 = 0$$

$$m_1, m_2 = \frac{3 \pm \sqrt{9+16}}{2} = 4, -1 \qquad y_g = c_1 e^{4x} + c_2 e^{-x}$$

$$a = -1, \quad m = a \qquad y_p = Axe^{-x} \qquad y_p' = Ae^{-x} - Axe^{-x}$$

$$y_p'' = -Ae^{-x} - Ae^{-x} + Axe^{-x} \qquad A = -\frac{1}{5}$$

$$y(x) = c_1 e^{4x} + c_2 e^{-x} - \frac{1}{5} xe^{-x}$$

Example: find G.S. $y'' - 10y' + 25y = e^{5x}$

$$y'' - 10y' + 25y = 0 \qquad m^2 - 10m + 25 = 0$$

$$m_1, m_2 = 5 \pm \sqrt{25 - 25} = 5$$

$$y_g = c_1 e^{5x} + c_2 x e^{5x} \quad \text{since} \quad m_1 = m_2 = a$$

$$y_p = Ax^2 e^{5x} \qquad y' = 2Axe^{5x} + 5Ax^2 e^{5x}$$

$$y'' = 2Ae^{5x} + 10Axe^{5x} + 10Axe^{5x} + 25Ax^2 e^{5x}$$

$$A = \frac{1}{2} \qquad y_p = \frac{1}{2} x^2 e^{5x} \qquad y(x) = c_1 e^{5x} + c_2 x e^{5x} + \frac{1}{2} x^2 e^{5x}$$

Case 2: $y'' + py' + qy = \sin bx \quad *$

$$y_p = A\sin bx + B\cos bx$$

$$y_p' = Ab\cos bx - Bb\sin bx$$

$$y_p'' = -Ab^2\sin bx - Bb^2\cos bx$$

$$\sin bx(-Ab^2 - pBb + qA) +$$

$$\cos bx(-Bb^2 + pAb + qB) = \sin bx$$

$$A(q - b^2) - pBb = 1 \quad Apb - B(b^2 - q) = 0$$

As before, the method breaks down if P.S. satisfies the homogeneous equation corresponding to * the procedure can be carried through by using:

$$y_p = x(A\sin bx + B\cos bx)$$

Example:Find P.S. $y'' - 3y' - 4y = 2\sin x$

$$y = A\sin x \Rightarrow y' = A\cos x,\ y'' = -A\sin x$$

$$-A\sin x - 3A\cos x - 4A\sin x = 2\sin x$$

$$(2 + 5A)\sin x + 3A\cos x = 0$$

$$c_1 \sin x + c_2 \cos x = 0$$

Since sin(*x*) and cos(*x*) are linearly independent (they are not multiple of each other), we must have $c_1 = c_2 = 0$, and hence $2 + 5A = 3A = 0$, which is impossible.

$$y_p = A\sin x + B\cos x$$

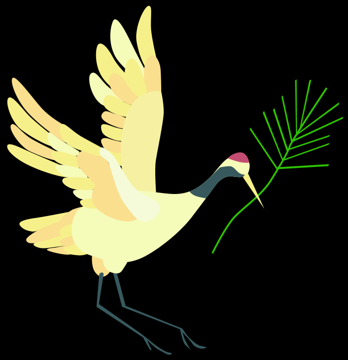
$$y_p' = A\cos x - B\sin x, \qquad y_p'' = -A\sin x - B\cos x$$

$$y'' - 3y' - 4y = 2\sin x$$

$$(-A\sin x - B\cos x) - 3(A\cos x - B\sin x) - 4(A\sin x + B\cos x) = 2\sin x$$

$$(-5A + 3B)\sin x + (-3A - 5B)\cos x = 2\sin x$$

$$-5A + 3B = 2, \qquad -3A - 5B = 0$$



$$A = -5/17, \qquad B = 3/17$$

$$y_p = \frac{-5}{17}\sin x + \frac{3}{17}\cos x$$

Euler Formula

$$e^{ix} = \cos x + i \sin x$$

Proof:

Maclaurin Series

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$e^{ix} = \cos x + i \sin x$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= \cos(x) + i \sin(x)$$

$$e^{i\pi} = -1$$