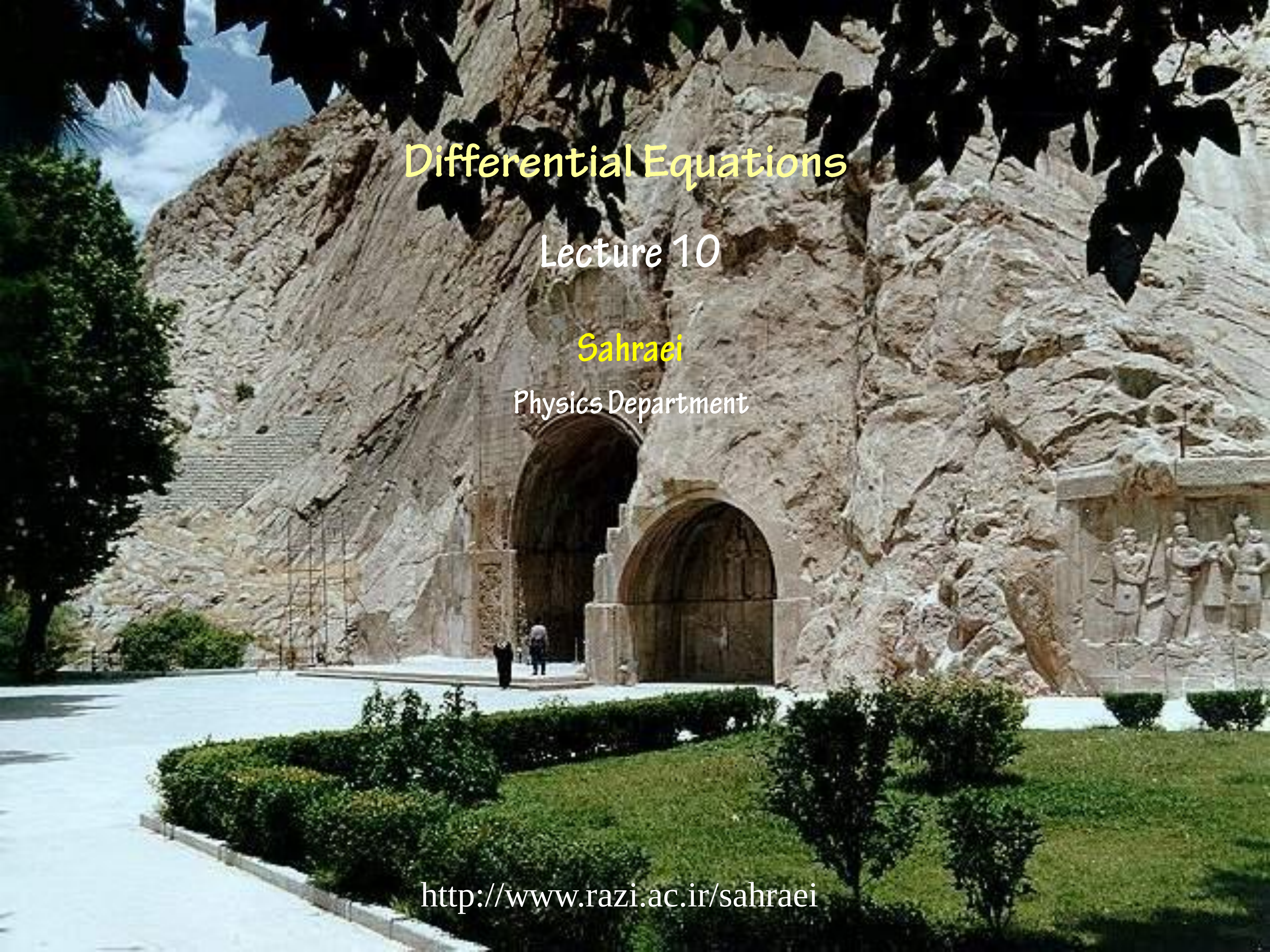

Differential Equations

Lecture 10

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Chapter 3

Second Order Linear Equations

A second order differential equation is *linear* if it can be written as:

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

$$y'' + P(x)y' + Q(x)y = R(x)$$

If *R(x)=0* the equation is homogeneous, otherwise it is non homogeneous.

Solving 2nd Order Linear Equation

Try reducing to first order equations.
This works for equations of the form:

Case 1-Dependent variable missing $f(x, y', y'') = 0$

$$y' = p \;,\; y'' = \frac{dp}{dx} \qquad f(x, p, \frac{dp}{dx}) = 0$$

Case 2-Independent variable missing $g(y, y', y'') = 0$

$$y' = p \;,\; y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy} \qquad g(y, p, p\frac{dp}{dy})$$

What about equations of the form:

$$F(x, y, y', y'') = 0\,?$$

Existence and Uniqueness of 2nd order equation

Existence: Does a differential equation have a solution?

Uniqueness: Does a differential equation have more than one solution? If yes, how can we find a solution which satisfies particular conditions?

Existence and Uniqueness of 2nd order equation

Theorem 1: **Let p(x), Q(x) and R(x) be continuous functions on a closed interval [a,b], if x_0 is any point in [a,b], and if y_0 and y'_0 are any numbers whatever, then equation**

$$y'' + p(x)y' + Q(x)y = R(x)$$

has one only one solution y(x) on the interval such that

$$y(x_0) = y_0 \quad , \quad y'(x_0) = y'_0$$

Example

Find the largest interval where

$$(x^2 - 1)y'' + 3xy' + \cos x\, y = e^x \qquad y(0) = 4,\ y'(0) = 5$$

is guaranteed to have a unique solution.

We first put it into standard form

$$y'' + 3x/(x^2 - 1)y' + (\cos x)/(x^2 - 1)\, y = e^x/(x^2 - 1)$$

$$y(0) = 4, \qquad y'(0) = 5$$

P, *Q*, and *R* are all continuous except at $x = -1$ and $x = 1$. The theorem tells us that there is a unique solution on [-1,1].

$$y'' + p(x)y' + Q(x)y = R(x) \quad (1)$$

$$y'' + p(x)y' + Q(x)y = 0 \quad (2)$$

Suppose that in some way we know that

$$y_g\left(x, c_1, c_2\right)$$

Is the genral solution of (2)

and that $y_p(x)$ is a fixed particular solution of (1)

If y(x) is any solution whatever of (1), then an easy calculation shows that *$y(x)-y_p(x)$* is the solution of (2).

$$y'' + p(x)y' + Q(x)y = R(x) \quad (1)$$

$$y(x) - y_p(x)$$

$$(y - y_p)'' + p(x)(y - y_p)' + Q(x)(y - y_p)$$

$$= \left[y'' + p(x)y' + Q(x)y\right] -$$
$$\left[y_p'' + p(x)y_p' + Q(x)y_p\right] = R(x) - R(x) = 0$$

$$y(x) = y_g(x, c_1, c_2) + y_p(x)$$

This argument proves the following theorem

Theorem 2: If y_g is the general solution of equation (2) and y_p is any particular solution of equation (1), then $y_g + y_p$ is the general solution of (1).

Theorem 3: If $y_1(x)$ and $y_2(x)$ are any two solution of (2), then

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

is also a solution for any constant c_1 and c_2.

PROOF

$$y'(x) = c_1 y_1'(x) + c_2 y_2'(x)$$

$$y''(x) = c_1 y_1''(x) + c_2 y_2''(x)$$

$$y'' + p(x) y' + Q(x) y = 0 \quad (2)$$

$$c_1 y_1''(x) + c_2 y_2''(x) + p(x)(c_1 y_1' + c_2 y_2') +$$
$$Q(x)(c_1 y_1 + c_2 y_2) =$$
$$c_1\left[y_1'' + p(x) y_1' + Q(x) y_1\right] +$$
$$c_2\left[y_2'' + p(x) y_2' + Q(x) y_2\right]$$
$$= c_1 \times 0 + c_2 \times 0 = 0$$

The solution below is commonly called a linear combination of the solutions y_1 (x) and y_2 (x).

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

Theorem 3: any linear combination of two solution of the homogeneous equation (2) is also a solution.

Problems page 83-1

By inspection, find the general solution of $y'' = e^x$

$$y(x) = y_g(x, c_1, c_2) + y_p(x)$$

$$y'' = 0 \rightarrow y' = c_1 \rightarrow y_g = c_1 x + c_2$$

$$y_p = e^x$$

$$y = c_1 x + c_2 + e^x$$

Problem page 83-2a - Find a P.S.

$$x^3 y'' + x^2 y' + xy = 1$$

$$x^3 y'' = c_1 \qquad x^2 y' = c_2 \qquad xy = c_3$$

$$y = \frac{c_3}{x} \to y' = -\frac{c_3}{x^2} \to y'' = \frac{2c_3}{x^3}$$

$$x^3\left(\frac{2c_3}{x^3}\right) + x^2\left(-\frac{c_3}{x^2}\right) + x\left(\frac{c_3}{x}\right) = 1$$

$$2c_3 - c_3 + c_3 = 1 \to c_3 = \frac{1}{2} \qquad y = \frac{1}{2x}$$

The general solution of the homogeneous equation

Definition: Two functions y_1 and y_2 are linearly dependent if there exist constants c_1 and c_2, not both zero, such that

$$c_1 y_1(x) + c_2 y_2(x) = 0$$

for all x in [a,b]. Note that this reduces to determining whether y_1 and y_2 are multiples of each other.

If the only solution to this equation is $c_1 = c_2 = 0$, then y_1 and y_2 are linearly independent.

Example: $y_1(x) = \sin 2x$ $y_2(x) = \sin x \cos x$

the linear combination $$c_1 \sin 2x + c_2 \sin x \cos x = 0$$

This equation is satisfied if we choose $c_1 = 1$, $c_2 = -2$, and hence y_1 and y_2 are linearly dependent.

Example $y_1 = x, \qquad y_2 = x^2$

$$c_1 x + c_2 x^2 = 0$$

$c_1 = c_2 = 0$, then y_1 and y_2 are linearly independent.

Theorem 4: **Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the homogeneous equation**

$$y'' + p(x)y' + Q(x)y = 0 \qquad (1)$$

on the interval [a,b] . Then

$$c_1 y_1(x) + c_2 y_2(x) \qquad (2)$$

is the general solution of equation (1) on [a,b], in the sense that every solution of (1) on this interval can be obtained from (2) by a suitable choice of the arbitrary constant c_1 and c_2.

Proof

$y(x)$ any solution of (1) on $[a, b]$

We must show that constants c_1 and c_2 for all x in [a,b] can be found so that

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

for some point x_0 in $[a, b]$ we can find c_1 and c_2 can be found so that

$$c_1 y_1(x_0) + c_2 y_2(x_0) = y(x_0)$$

$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = y'(x_0)$$

Consider the linear system (in matrix form) $\boldsymbol{A\,c = B}$

$$c_i = \frac{\det(A_i)}{\det A}, \quad for\ \mathrm{i} = 1,\ldots,n$$

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y(x_0) \\ y'(x_0) \end{pmatrix}$$

$$c_1 = \frac{y y_2' - y' y_2}{y_1 y_2' - y_1' y_2} \qquad c_2 = \frac{y y_1' - y' y_1}{y_1 y_2' - y_1' y_2}$$

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 \neq 0$$ **Wronskian of y_1 & y_2**

The Wronskian of two linearly independent solution of (1) is not identically zero.

If and only if their Wronskian $W(y_1, y_2)$ is zero they are linearly dependent .

Lemma 1: **If $y_1(x)$, $y_2(x)$ are two solution of equation (1) on [a,b], then their Wronskian $W=W(y_1,y_2)$ is either identically zero or never zero on [a,b].**

Lemma, a proven statement used as a stepping-stone toward the proof of another statement

PROOF

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2$$

$$W'(y_1, y_2) = y_1 y_2'' + y_1' y_2' - y_1'' y_2 - y_1' y_2'$$

$$= y_1 y_2'' - y_1'' y_2$$

Next, since y_1 and y_2 are both solutions of (1), we have:

$$y'' + p(x) y' + Q(x) y = 0 \qquad (1)$$

$$y_1'' + p y_1' + Q y_1 = 0 \qquad y_2'' + p y_2' + Q y_2 = 0$$

$$y_2\left(y_1'' + py_1' + Qy_1\right) = 0 \quad (1)$$

$$y_1\left(y_2'' + py_2' + Qy_2\right) = 0 \quad (2)$$

$$(2) - (1) \rightarrow (y_1 y_2'' - y_2 y_1'') + P(y_1 y_2' - y_2 y_1') = 0$$

$$\frac{dW}{dx} + PW = 0 \qquad \frac{dW}{W} = -P(x)dx$$

$$\ln W = -\int P(x)dx + c$$

$$W = e^{-\int P(x)dx + c} = Ce^{-\int P(x)dx} \quad \text{G.S.}$$

Since the exponential factor is never zero, the proof is completed.

Lemma 2: **If $y_1(x)$ and $y_2(x)$ are two solution of equation (1) on [a,b], then they are linearly dependent on this interval if and only if their Wronskian $W(y_1,y_2)=y_1y'_2-y_2y'_1$ is identically zero.**

PROOF $$y_2 = cy_1 \rightarrow y_2' = cy_1'$$

$$W(y_1, y_2) = y_1y_2' - y_1'y_2 = y_1cy_1' - y_1'cy_1 \equiv 0$$

Since the Wronskian is identically zero on [a,b], we can divided it by y_1^2 to get:

$$\frac{y_1y_2' - y_2y_1'}{y_1^2} = 0 \rightarrow (y_2 / y_1)' = 0$$

$$\frac{y_2}{y_1} = k \rightarrow y_2(x) = ky_1(x)$$

Problems page 87-1: **show that e^x and e^{-x} are linearly independent solution of** $y'' - y = 0$ **on any interval.**

$$y_1 = e^x \rightarrow y_1' = e^x \rightarrow y_1'' = e^x$$

$$y_1'' - y_1 = e^x - e^x = 0$$

$$y_2 = e^{-x} \rightarrow y_2' = -e^{-x} \rightarrow y_2'' = e^{-x}$$

$$y_2'' - y_2 = e^{-x} - e^{-x} = 0$$

$$\frac{y_1}{y_2} = e^{2x} \neq k$$

$$W(y_1, y_2) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^x e^{-x} - e^x e^{-x} = -2$$

Example: show that $y = c_1 \sin x + c_2 \cos x$

is general solution of $y'' + y = 0$ on any interval, and find the P.S. for which y(0)=2 and y′(0)=3.

$$y_1 = \sin x, \quad y_2 = \cos x \rightarrow y_1 / y_2 = tgx \rightarrow W \neq 0$$

$$y_1'' + y_1 = 0 \qquad y_2'' + y_2 = 0$$

$$W(y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1$$

$$c_1 \sin 0 + c_2 \cos 0 = 2$$

$$c_1 \cos 0 - c_2 \sin 0 = 3$$

$$\Rightarrow y = 3\sin x + 2\cos x$$

Problems page 87-3: **Show that** $y = c_1 e^{x} + c_2 e^{2x}$ **is general solution of**

$$y'' - 3y' + 2y = 0$$

On any interval, and find the particular solution for which y(0)=-1 and y′(0)=1.

$$y' = c_1 e^{x} + 2c_2 e^{2x}, \quad y'' = c_1 e^{x} + 4c_2 e^{2x}$$

$$c_1 e^{x} + 4c_2 e^{2x} - 3(c_1 e^{x} + 2c_2 e^{2x}) + 2(c_1 e^{x} + c_2 e^{2x}) =$$

$$= \underline{c_1 e^{x}} + 4c_2 e^{2x} - \underline{3c_1 e^{x}} - 6c_2 e^{2x} + \underline{2c_1 e^{x}} + 2c_2 e^{2x} = 0$$

$$y_1 = e^x\ ,\ \ y_2 = e^{2x} \qquad y_1' = e^x\ ,\ \ y_2' = 2e^{2x}$$

$$W(y_1, y_2) = \begin{vmatrix} e^x & \mathrm{e}^{2\mathrm{x}} \\ e^x & 2\mathrm{e}^{2\mathrm{x}} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x}$$

$$y = c_1 e^x + c_2 e^{2x}$$

$$y(0) = -1 \rightarrow c_1 + c_2 = -1$$

$$c_1 = -3$$

$$y' = c_1 e^x + 2c_2 e^{2x}$$

$$c_2 = 2$$

$$y'(0) = 1 \rightarrow c_1 + 2c_2 = 1$$

$$y = -3e^x + 2e^{2x}$$

The use of a known solution to find another

As we have seen , it is easy to write down the general solution of the homogeneous equation

$$y'' + p(x)y' + Q(x)y = 0 \quad (1)$$

Whenever we know two linearly independent solution $y_1(x)$ and $y_2(x)$. But how do we find y_1 and y_2?

$$y_1(x) \rightarrow \textit{is a known nonzero solution}$$

$$cy_1(x) \rightarrow \textit{is also solution for any}\ \text{constant}\ c$$

$$v(x)\ \textit{an unknown function} \rightarrow c$$

$$y_2 = vy_1\ \textit{will be solution of}\ (1)$$

We assume, then that $y_2 = vy_1$ is a solution of (1), so that

$$y_2'' + Py_2' + Qy_2 = 0$$

$$y_2 = vy_1$$

$$y_2' = vy_1' + v'y_1$$

$$y_2'' = vy_1'' + 2v'y_1' + v''y_1$$

$$v(y_1'' + Py_1' + Qy_1) + v''y_1 + v'(2y_1' + Py_1) = 0$$

$$v''y_1 + v'(2y_1' + Py_1) = 0$$

$$\frac{v''}{v'} = -2\frac{y_1'}{y_1} - P \qquad \int \frac{v''}{v'} = -2\int \frac{y_1'}{y_1} - \int P dx$$

$$\ln v' = -2\ln y_1 - \int P dx \qquad \ln v' y_1{}^2 = -\int P dx$$

$$v' = \frac{1}{y_1{}^2} e^{-\int P dx} \qquad v = \int \frac{1}{y_1{}^2} e^{-\int P dx} dx$$

All that remain is to show that y_1 and $y_2 = vy_1$, where v is given by above equation actually are linearly independent as claimed;

Problems page 90-1

$$W(y_1, y_2) \neq 0$$

$$v = \int \frac{1}{{y_1}^2} e^{-\int P dx} dx \quad y_2 = v y_1 = y_1 \int \frac{1}{{y_1}^2} e^{-\int P dx} dx$$

$$y_2' = y_1' \int \frac{1}{{y_1}^2} e^{-\int P dx} dx + y_1 \left(\frac{1}{{y_1}^2} e^{-\int P dx} \right)$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_1 \int \frac{1}{{y_1}^2} e^{-\int P dx} dx \\ y_1' & y_1' \int \frac{1}{{y_1}^2} e^{-\int P dx} dx + \frac{1}{y_1} e^{-\int P dx} \end{vmatrix} = e^{-\int P dx} \neq 0$$

Example: $y_1 = x$ **is a solution of**

Find G.S.? $$x^2 y'' + xy' - y = 0$$

$$y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0 \qquad v = \int \frac{1}{y_1^2} e^{-\int P dx} dx$$

$$P(x) = \frac{1}{x}$$

$$v = \int \frac{1}{x^2} e^{-\int 1/x dx} dx = \int \frac{1}{x^2} e^{-\ln x} dx$$

$$= \int x^{-3} dx = \frac{x^{-2}}{-2} \qquad y_2 = v y_1 = \left(\frac{x^{-2}}{-2}\right) x = -\frac{1}{2} x^{-1}$$

$$y_g = c_1 x - \frac{c'}{2} x^{-1} = c_1 x + c_2 x^{-1}$$

Example: $y'' - 4y' - 12y = 0, \qquad y_1 = e^{6x}$

$$v = \int \frac{1}{y_1^{2}} e^{-\int P\,dx}\,dx = \int \frac{1}{e^{12x}} e^{4\int dx}\,dx$$

$$= \int e^{-12x} e^{4x}\,dx = -\frac{1}{8} e^{-8x}$$

$$y_2 = v y_1 = -\frac{1}{8} e^{-8x} e^{6x} = -\frac{1}{8} e^{-2x}$$

$$y_g = c_1 e^{6x} - \frac{c'}{8} e^{-2x} = c_1 e^{6x} + c_2 e^{-2x}$$

Thanks for listening

Any Question ?