Differential Equations

Lecture 1

Sahraei

Physics Department

http://www.razi.ac.ir/sahraei

References: منابع:

Differential Equations with Applications and Historical Notes, George F. Simmons

معادلات دیفرانسیل و کاربردهای آنها

جرج ف سیمونز ترجمه بابایی ، میامئی

معادلات دیفرانسیل دکتر مسعود نیکوکار

اینترنت

Assessment

- **Mid semester exam : 50%**
- **Final exam: 50%**

What is differential equations?

Equation: Equations describe the relations between the dependent and independent variables.
An equal sign "=" is required in every equation.

A differential equation is an equation that defines a relationship between a function and one or more derivatives of that function.

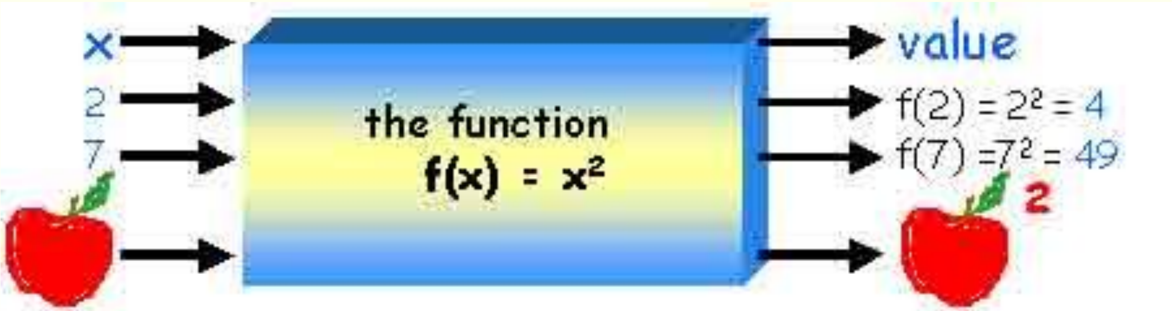
تعریف: هر رابطه بین تابع و متغیر مستقل و مشتقات تابع نسبت به متغیر مستقل را یک معادلات دیفرانسیل می نامیم.

A *differential equation* is an algebraic relation between variables that includes the rates of change of the variables as well as their instantaneous values.

$y = f(x)$ a given function

dy/dx its derivative

در ریاضیات، یک تابع رابطه‌ای است که هر متغیر دریافتی خود را به فقط یک خروجی نسبت می‌دهد. علامت استاندارد خروجی یک تابع f به همراه ورودی آن، x می‌باشد یعنی f(x). به مجموعه ورودی‌هایی که یک تابع می‌تواند داشته باشد دامنه و به مجموعه خروجی‌هایی که تابع می‌دهد برد می‌گویند.



Example

If population growth rate of a species of animals is proportional to the number of animals at the moment. Then growth rate can be modelled by the following differential equation:

$$\frac{dy}{dt} = ky(t)$$

t is the independent variable (time)
y is the dependent variable (number of animals)
k is the parameter

$\frac{dy}{dt} = ky(t)$ **is a differential equation as it contains the function y(t) and its derivative** $\frac{dy}{dt}$

Let *y* be some function of the independent variable *t*. Then following are some differential equations relating *y* to one or more of its derivatives.

The equation

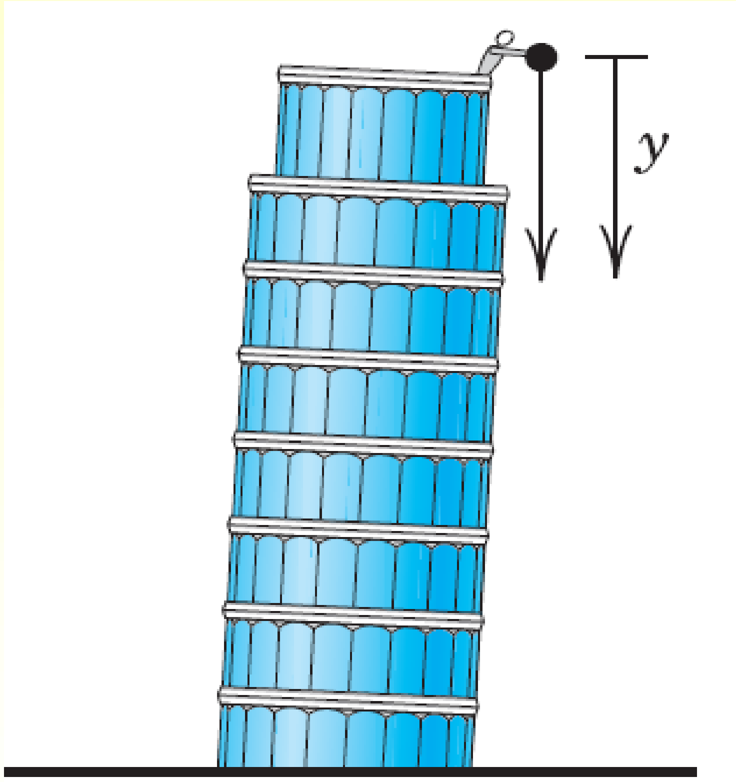
$$\frac{\partial}{\partial t} y(t) = t^2\, y(t)$$

states that the first derivative of the function *y* equals the product of t^2 and the function *y* itself.

Applications of differential equations

Differential equations play an extremely important and useful role in applied math, engineering, and physics, and much mathematical and numerical machinery has been developed for the solution of differential equations.

Freely Falling Objects



Falling stone

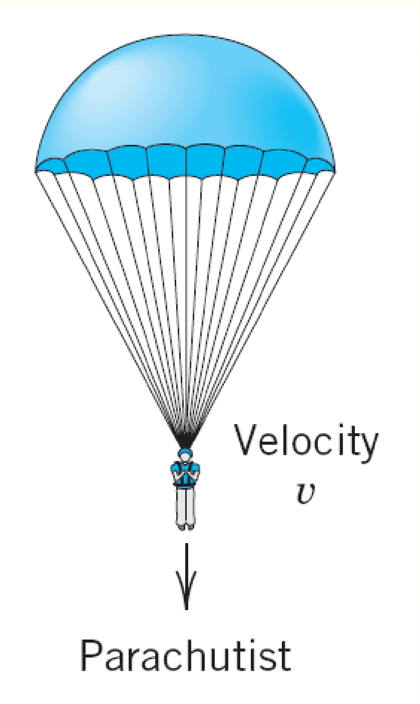
$$F = ma$$

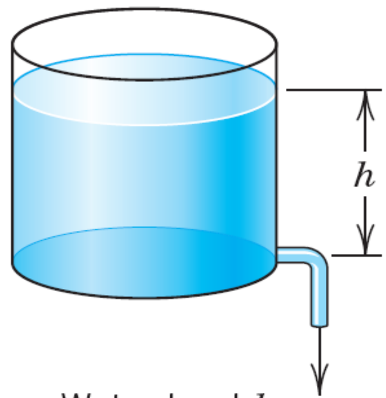
$$F = m\frac{dv}{dt}$$

$$m\frac{d^2y}{dt^2} = mg$$

$$\frac{d^2y}{dt^2} = g$$

$$m\frac{d^2y}{dt^2} = mg - k\frac{dy}{dt}$$

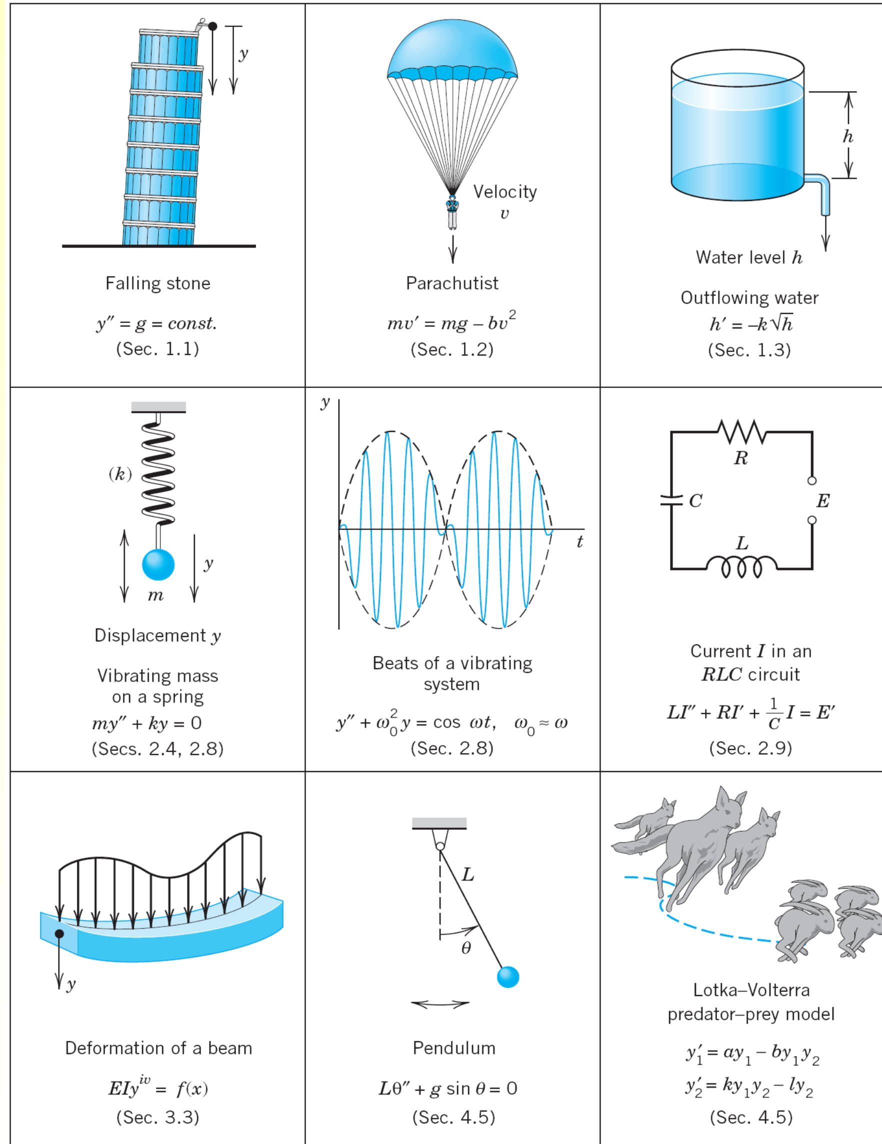




Water level h

Outflowing water

$h' = -k\sqrt{h}$



پدیده فیزیکی وقتی با علائم ریاضی بیان شود نتیجه یک معادله دیفرانسیل می باشد.

Types of differential equations

Odinary Differential Equations

An Ordinary Differential Equation is a differential equation that depends on only one independent variable.

For example

$$\frac{dy}{dt} = ky(t)$$

is an Odinary Differential Equation because y (the dependent variable) depends only on t(the independent variable)

Partial Differential Equations

A Partial Differential Equation is differential equation in which the dependent variable depends on two or more independent variables.

For example $w = f(x, y, z, t)$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$$ Laplace' s equation

$$a^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \frac{\partial w}{\partial t}$$ the heat equation

$$a^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \frac{\partial^2 w}{\partial t^2}$$ the wave equation

Order of a Differential Equation

The order of a differential is the order of the highest derivative entering the equation.

بالاترین مرتبه مشتق موجود در معادله دیفرانسیل را مرتبه معادله دیفرانسیل گوییم.

For example

The equation $m\frac{d^2x}{dt^2} = -kx$ **is called a second-order differential equation because it involves second derivatives.**

Degree of a Differential Equation

Degree: The degree of a differential equation is the power of the *highest derivative* term.

توان بالاترین مشتق موجود در معادله دیفرانسیل را درجه معادله می نامند.

$$y'' + xy^2 (dy/dx)^3 = e^x$$

$$\text{(i)}\ \left(\frac{d^2y}{dx^2}\right)^3 = 2x^2 + 7\sqrt{x}$$

(i) Order = 2 degree = 3

$$\text{(ii)}\ 3\left(\frac{dy}{dx}\right)^2 = \sin 2x$$

(ii) Order =1 degree = 2

$$\text{(iii)}\ y = x\frac{dy}{dx} + c\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

(iii) Order =1 degree = 2

Linear Differential Equation

A first-order differential equation is linear if it can be written in the form

$$\frac{dy}{dx} = a(x)y + b(x)$$

where *a(x)* and *b(x)* are arbitary functions of *x*.

For example

$$\frac{dy}{dx} = x^2 y + \cos(x)$$ **is a first-order linear differential equation**

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + x^2 y = 5$$

A linear differential equation of order *n* is a differential equation written in the following form:

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}}+\dots+a_1(x)\frac{dy}{dx}+a_0(x)y= f(x)$$

where is not the zero function.

Linear: A differential equation is called linear if there are no multiplications among dependent variables and their derivatives. In other words, all coefficients are functions of independent variables.

Nonlinear Differential Equation

It is a differential equation whose right hand side is not a linear function of the dependent variable.

For example

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P$$

$$\frac{d^3y}{dx^3} + xy^2\left(\frac{dy}{dx}\right)^2 = 5$$

Momentum Equations in Spherical Coordinates

$$f = f(x, y, z; t)$$

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - 2\Omega w\cos\phi + F_{rx}$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi + F_{ry}$$

$$\frac{dw}{dt} - \frac{u^2 + v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + 2\Omega u\cos\phi - g + F_{rz}$$

Because they are nonlinear (that is, they are quadratic in the dependent variables) they are difficult to handle in theoretical analyses.