ANALYTICAL MECHANICS 1

Lecture 9

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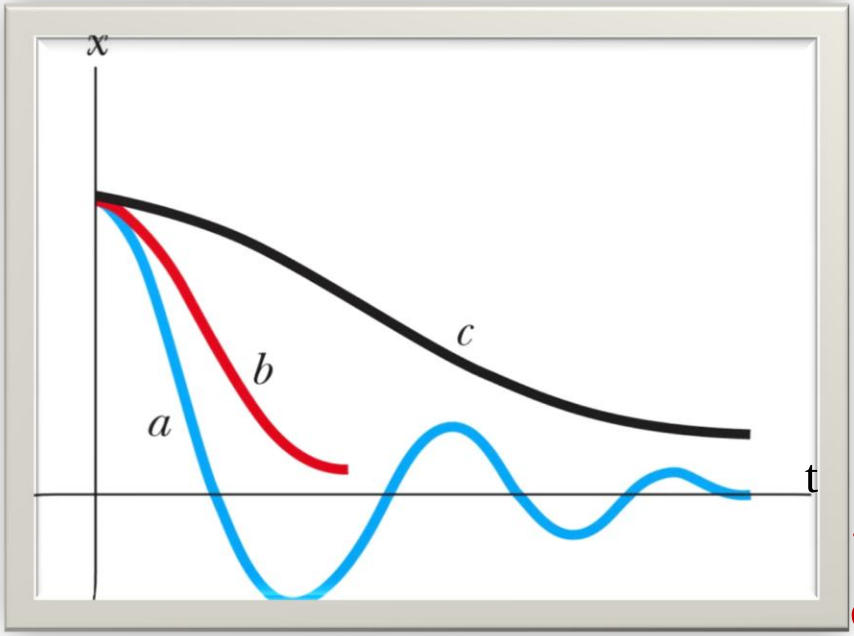
Types of Damping



a) an underdamped oscillator

The restoring force is large compared to the damping force.
The system oscillates with decaying amplitude.

b) a critically damped oscillator

The restoring force and damping force are comparable in effect.
The system can not oscillate; the amplitude dies away exponentially

c) an overdamped oscillator

The damping force is much stronger than the restoring force.
The amplitude dies away as a modified exponential

Mechanical Suspensions

$$c^2 = 4km_{crit} \quad c^2 - 4km = 4k(m_{crit} - m)$$

$$m < m_{crit} \rightarrow overdamped$$

$$m > m_{crit} \rightarrow underdamped$$

Energy consideration

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = \dot{x}(m\ddot{x} + kx)$$

$$m\ddot{x} + c\dot{x} + kx = 0$$ MOTION EQUATION

$$m\ddot{x} + kx = -c\dot{x} \qquad \frac{dE}{dt} = -c\dot{x}^2$$

$$\frac{dE}{dt} < 0 \rightarrow total\ \ energy\ \ contiunally\ \ decreases$$

Examples

Example 3.4- An automobile suspension system is critically damped, and its period of free oscillation with no damping is one second. If the system is initially displaced by an amount x_0, and released with zero initial velocity, find the displacement at t=1s.

$$c^2 - 4km = 0 \qquad \frac{c^2}{4m^2} - \frac{k}{m} = 0 \qquad \gamma = c/2m$$

$$\omega_0 = (k/m)^{1/2} \qquad \gamma = \omega_0 = \frac{2\pi}{T_0} = 2\pi\ s^{-1}$$

$$x(t)=(At+B)e^{-\gamma t} \qquad t=0,\ x_0=B,\ \dot{x}_0=0$$

$$\dot{x}(t)=(A-\gamma B-\gamma At)e^{-\gamma t} \qquad A=\gamma B=\gamma x_0$$

$$x(t)=x_0(1+\gamma t)e^{-\gamma t} \qquad x(t)=x_0(1+2\pi t)e^{-2\pi t}$$

$$For\ t=1s \quad x(1s)=x_0(1+2\pi)e^{-2\pi}=0.0136x_0$$

Example 3.5- The frequency of a damped harmonic oscillator is one-half the frequency of the same oscillator with no damping. Find the ratio of the maxima of successive oscillations.

$$\omega_d=\frac{1}{2}\omega_0 \qquad \omega_d=(\omega_0{}^2-\gamma^2)^{1/2} \qquad \frac{\omega_0{}^2}{4}=\omega_0{}^2-\gamma^2$$

$$\gamma^2 = \frac{3}{4}\omega_0^{\ 2} \qquad \gamma = (\frac{3}{4})^{1/2}\omega_0$$

$$Ae^{-\gamma t} \rightarrow Ae^{-\gamma(t+T_d)} \quad \frac{Ae^{-\gamma(t+T_d)}}{Ae^{-\gamma t}} = e^{-\gamma T_d}$$

$$e^{-\gamma T_d} = e^{-(3/4)^{1/2}\omega_0 \frac{4\pi}{\omega_0}}$$

$$= e^{-2\pi\sqrt{3}} = e^{-10.88} = 0.00002$$ i.e. highly damped oscillator

Example 3.6- The terminal speed of a baseball in free fall is 30 m/s. Assuming a linear air drag, calculate the effect of air resistance on a simple pendulum using a baseball as the bob.

$$v_t = \frac{mg}{c_1} = 30m/s \quad \gamma = \frac{c_1}{2m} = \frac{mg/30}{2m} = 0.163s^{-1}$$

$$e^{-1} \rightarrow \gamma^{-1} = 6.31s \quad \omega_0 = \sqrt{\frac{g}{l}} \quad \omega_d = (\omega_0^{\ 2} - \gamma^2)^{1/2}$$

$$T_d = 2\pi(\omega_0^{\ 2} - \gamma^2)^{-1/2}$$

$$T_d = 2\pi\left(\frac{g}{l} - 0.0265s^{-2}\right)^{-1/2} = 2.00268s$$

$$T_0 = 2\pi\sqrt{\frac{l}{g}} \quad 2 = 2\pi\sqrt{\frac{l}{g}} \quad \pi^2 = \frac{g}{l} \rightarrow l = 1m$$

Natural Frequency

*When a system is **disturbed**, it will oscillate with a frequency which is called the **natural frequency** (f_o) of the system.*

e.g. for a mass-spring system :

$$f_o = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Forced Oscillation

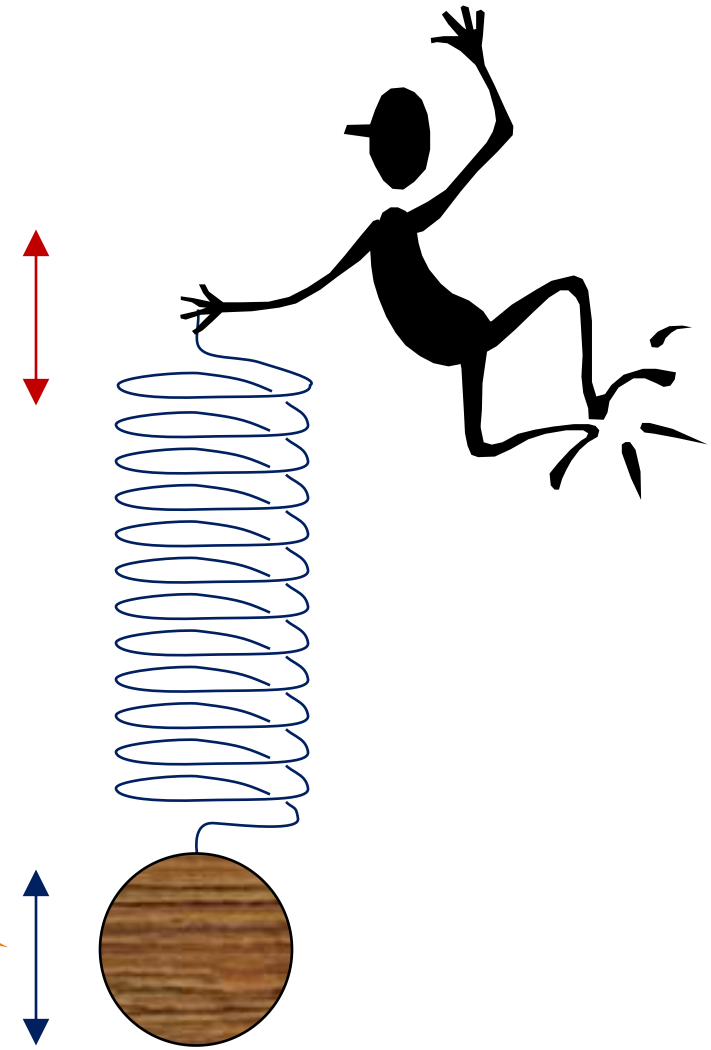
*When a system is disturbed by a **periodic driving force** and then oscillate, this is called **forced oscillation.***

*Note : The system will oscillate with **its natural frequency** (f_o) which is **independent of** the frequency of the driving force.*

Example (Mass-Spring System)

Periodic driving force of freq. f

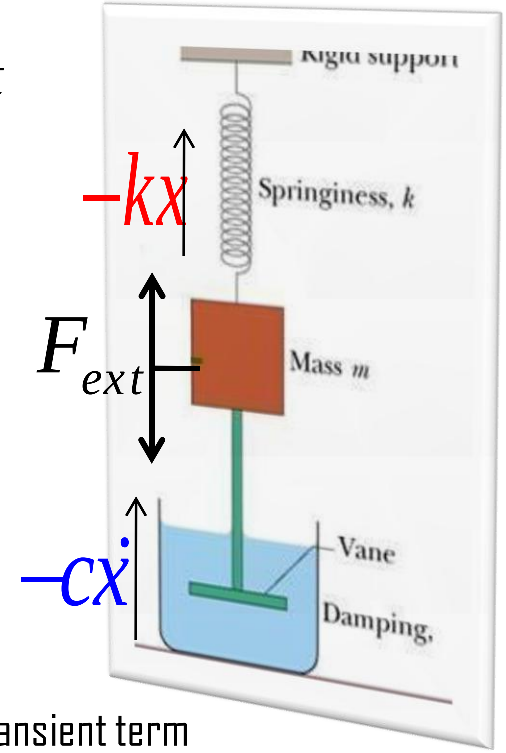
Oscillating with natural freq. f_o



Forced Harmonic Motion. Resonance

$$F = Ma \qquad m\ddot{x} = -kx - c\dot{x} + F_{ext}$$

$$F_{ext} = F_0 e^{i\omega t} \qquad F_{ext} = F_0 \cos\omega t$$



$$m\ddot{x} + c\dot{x} + kx = F_{ext}$$

$$x(t) = x_p + x_g$$

$$x_g = e^{-\gamma t}(A\cos\omega_d t + B\sin\omega_d t)$$ Transient term

$$x_p = A\sin\omega t + B\cos\omega t$$

$$\dot{x}_p = A\omega\cos\omega t - B\omega\sin\omega t$$

$$\ddot{x}_p = -A\omega^2 \sin\omega t - B\omega^2 \cos\omega t$$

$$\sin\omega t\left[A(k - m\omega^2) - c\omega B\right] + \cos\omega t\left[B(k - m\omega^2) + c\omega A\right]$$

$$= F_0 \cos\omega t$$

$$A(k - m\omega^2) - c\omega B = 0 \qquad B(k - m\omega^2) + c\omega A = F_0$$

$$A = \frac{c\omega F_0}{(k - m\omega^2)^2 + \omega^2 c^2} \qquad B = \frac{(k - \omega^2 m)F_0}{(k - \omega^2 m)^2 + \omega^2 c^2}$$

$$x_p = \frac{c\omega F_0}{(k - m\omega^2)^2 + \omega^2 c^2}\sin\omega t + \frac{(k - \omega^2 m)F_0}{(k - \omega^2 m)^2 + \omega^2 c^2}\cos\omega t$$

$$x_p = \frac{F_0 c\omega}{(k - m\omega^2)^2 + \omega^2 c^2}\left[\sin\omega t + \frac{(k - \omega^2 m)}{c\omega}\cos\omega t\right]$$

$$tg\,\varphi = \frac{\omega c}{k - m\omega^2}$$

$$x_p = \frac{F_0 c\omega}{(k - m\omega^2)^2 + \omega^2 c^2}\left[\sin\omega t + \frac{\cos\varphi}{\sin\varphi}\cos\omega t\right]$$

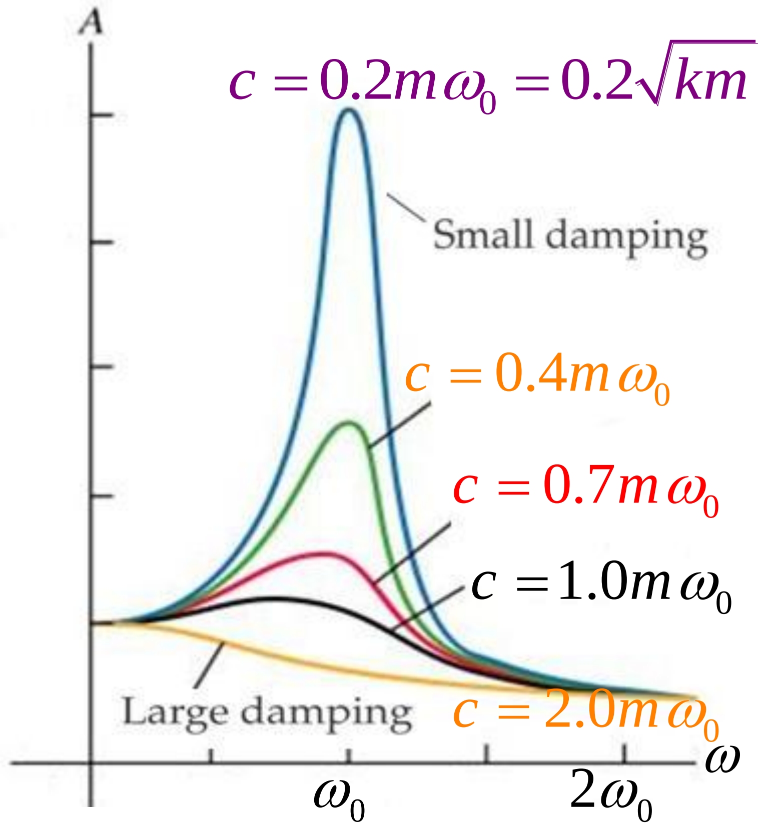
$$x_p = \frac{F_0 c\omega}{\sin\varphi\left[(k - m\omega^2)^2 + \omega^2 c^2\right]}\cos(\omega t - \varphi)$$

$$x_p = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \omega^2 c^2}}\cos(\omega t - \varphi)$$

$$A(\omega) = \frac{F_0}{\left[(k - m\omega^2)^2 + \omega^2 c^2\right]^{1/2}}$$

$$\omega_0 = (k/m)^{1/2} \qquad \gamma = c/2m$$

$$A(\omega) = \frac{F_0/m}{\left[(\omega_0{}^2 - \omega^2)^2 + 4\gamma^2\omega^2\right]^{1/2}}$$



Greater damping (larger c):

- Peak becomes broader
- Peak becomes less sharp
- Peak shifts toward lower frequencies

If $c > \sqrt{2km}$, peak disappears completely

$$\gamma \geq \frac{\omega_0}{\sqrt{2}}$$

The amplitude increases with decreased damping
The curve broadens as the damping increases
The shape of the resonance curve depends on c

$$\frac{dA(\omega)}{d\omega} = 0$$

$$\frac{\frac{F_0}{m}\left(-\frac{1}{2}\right)\left[2(-2\omega)(\omega_0^{\ 2} - \omega^2) + 8\gamma^2\omega\right]}{(\omega_0^{\ 2} - \omega^2 + 4\gamma^2\omega^2)^{3/2}} = 0$$

$$-4\omega(\omega_0^{\ 2} - \omega^2) + 8\gamma^2\omega = 0$$

$$\omega\left[-4(\omega_0^{\ 2} - \omega^2) + 8\gamma^2 = 0\right]$$

$$\omega_r^{\ 2} = \frac{4\omega_0^{\ 2} - 8\gamma^2}{4} = \omega_0^{\ 2} - 2\gamma^2$$

$$\omega_d^{\ 2} = \omega_0^{\ 2} - \gamma^2 \qquad \omega_r^{\ 2} = \omega_d^{\ 2} - \gamma^2$$

$$\omega_r = (\omega_d{}^2 - \gamma^2)^{1/2}$$

$$A(\omega) = \frac{F_0 / m}{\left[(\omega_0{}^2 - \omega^2)^2 + 4\gamma^2\omega^2\right]^{1/2}} \qquad \gamma^2 = \frac{\omega_0{}^2}{2}$$

$$A(\omega) = \frac{F_0 / m}{\left[(\omega_0{}^2 - \omega^2)^2 + 2\omega_0\omega^2\right]^{1/2}} = \frac{F_0 / m}{(\omega_0{}^4 + \omega^4)^{1/2}}$$

$$\omega \uparrow \rightarrow A \downarrow$$

Amplitude of Oscillation at the Resonance Peak

$$A_{\max} = \frac{F_0 / m}{\left[(\omega_0{}^2 - \omega_d{}^2 + \gamma^2)^2 + 4\gamma^2(\omega_d{}^2 - \gamma^2)\right]^{1/2}}$$

$$A_{max} = \frac{F_0 / m}{\left[(\omega_0{}^2 - \gamma^2 - \omega_d{}^2 + 2\gamma^2)^2 + 4\gamma^2(\omega_d{}^2 - \gamma^2)\right]^{1/2}}$$

$$= \frac{F_0 / m}{2\gamma\omega_d} = \frac{F_0}{c\,\omega_d}$$

$$\gamma = c / 2m$$

Case of weak damping $\gamma << \omega_0$

$$\omega_d{}^2 = \omega_0{}^2 - \gamma^2 \qquad \omega_d \cong \omega_0 \qquad A_{max} = \frac{F_0}{c\,\omega_0}$$

Resonance

When a system is disturbed by a periodic driving force which frequency is ***equal to the natural frequency (fo)*** *of the system, the system will oscillate with* ***LARGE amplitude.***
Resonance *is said to occur.*

If the driving frequency is close to the natural frequency, the amplitude can become quite large, especially if the damping is small. This is called resonance.

This system will show the property of *resonance*. The oscillation amplitude will depend on the driving frequency ω, and will have its maximum value when: $\omega = \omega_0 = \sqrt{k/m}$ i.e., when the system is driven at its natural resonant frequency ω_0.

Example 1

Breaking Glass

System : ***glass***

Driving Force :

sound wave

Example 2

Collapse of the Tacoma Narrows suspension bridge in America in 1940

System : ***bridge***

Driving Force :
strong wind

Sharpness of the Resonance.

Case of weak damping $\gamma << \omega_0$ $(\omega_0 \simeq \omega)$

$$A(\omega) = \frac{F_0/m}{\left[(\omega_0{}^2 - \omega^2)^2 + 4\gamma^2\omega^2\right]^{1/2}}$$

$$\omega_0{}^2 - \omega^2 = (\omega_0 + \omega)(\omega_0 - \omega) = (2\omega_0)(\omega_0 - \omega)$$

$$\gamma\omega \simeq \gamma\omega_0 \qquad A(\omega) = \frac{F_0/m}{\sqrt{4\omega_0{}^2(\omega_0 - \omega)^2 + 4\gamma^2\omega_0{}^2}}$$

$$A(\omega) = \frac{F_0/m}{\sqrt{4\omega_0^{\ 2}(\omega_0-\omega)^2+4\gamma^2\omega_0^{\ 2}}} = \frac{F_0/m}{2\omega_0\sqrt{(\omega_0-\omega)^2+\gamma^2}}$$

$$= \frac{F_0/2\omega_0 m}{\sqrt{(\omega_0-\omega)^2+\gamma^2}} = \frac{cA_{max}/2m}{\sqrt{(\omega_0-\omega)^2+\gamma^2}} = \frac{\gamma A_{max}}{\sqrt{(\omega_0-\omega)^2+\gamma^2}}$$

$$if \quad (\omega_0-\omega)^2 = \gamma^2$$

$$|\omega_0-\omega| = \gamma$$

$$\omega = \omega_0 \pm \gamma$$

$$A_{max} = \frac{F_0}{c\,\omega_0}$$

$$A^2 = \frac{1}{2}A_{max}^{\ \ 2}$$

Quality Factor

There is a quantitative measure of how sharp the resonance is

"Sharpness" of resonance peak described by quality factor (Q)

the resonance peak frequency ω_o

$$Q = \frac{\omega_d}{2\gamma}$$

Case of weak damping

$$Q \simeq \frac{\omega_0}{2\gamma}$$

$$\Delta\omega = 2\gamma \simeq \frac{\omega_0}{Q}$$

$$\omega = 2\pi f$$

Bandwidth, BW = difference between the two half-power frequencies

$$\frac{\Delta\omega}{\omega_0} = \frac{\Delta f}{f_0} \simeq \frac{1}{Q}$$

The higher the Q, the smaller the bandwidth

The **quality factor** of a resonant circuits is the ratio of its resonant frequency to its bandwidth

$$Q = \pi \frac{\text{decay time}}{\text{period}} = \pi \frac{1/\gamma}{2\pi/\omega_o} = \frac{\omega_o}{2\gamma}.$$

The ***width*** of the resonance curves depends on c, i.e., on the amount of damping. Wider curves with smaller resonance maxima correspond to more damping and to larger values of c.