ANALYTICAL MECHANICS 1

Lecture 8

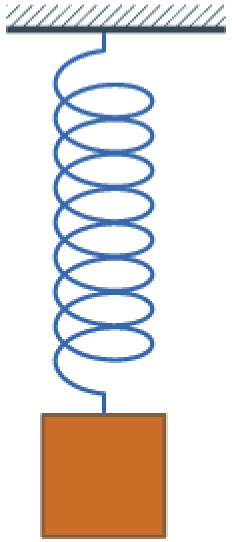
Sahraei

Physics Department,

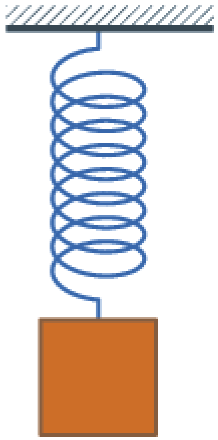
Razi University

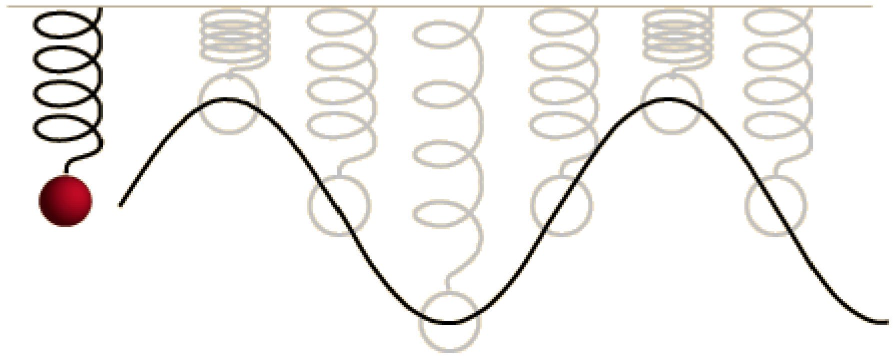
http://www.razi.ac.ir/sahraei

Simple Harmonic Motion



Damped Harmonic Motion





Energy Consideration in Harmonic Motion

$$F_x = -kx \quad \text{Restoring Force} \quad F_{ext} = -F_x = kx$$

$$W = \int_0^x F_{ext}\,dx = \int_0^x kx\,dx = \frac{1}{2}kx^2$$

$$V(x) = \frac{1}{2}kx^2 \quad F_x = -\frac{dV}{dx} = -kx$$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}}(E - \frac{k}{2}x^2)^{1/2}$$

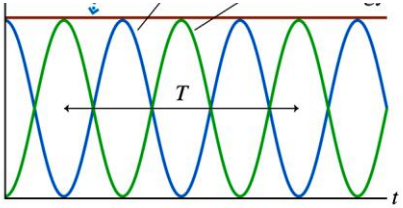
$$\int dt = \int \frac{dx}{\pm\sqrt{\frac{2}{m}}(E - \frac{k}{2}x^2)^{1/2}}$$

$$\int dt = \int \frac{dx}{\pm\sqrt{\frac{2}{m}}\sqrt{\frac{k}{2}}(\frac{2E}{k} - x^2)^{1/2}}$$

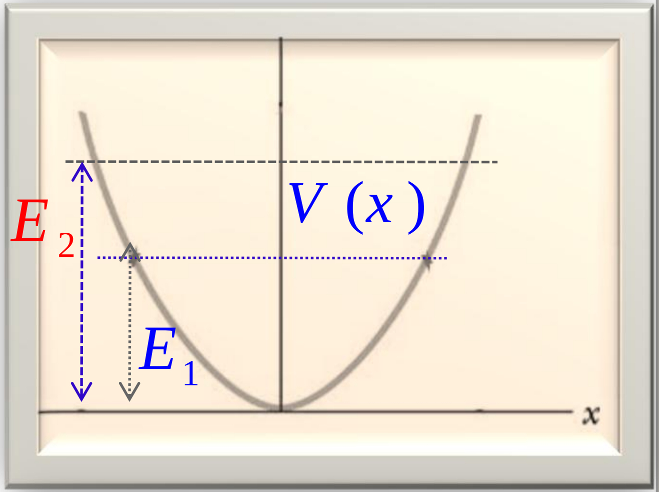
$$\sqrt{\frac{k}{m}}\int dt = \int \frac{dx}{\pm(\frac{2E}{k} - x^2)^{1/2}}$$

$$x = \sqrt{\frac{2E}{k}}\cos\theta \qquad dx = -\sqrt{\frac{2E}{k}}\sin\theta d\theta$$

$$\omega_0 t = \int \frac{\sqrt{\frac{2E}{k}}\sin\theta d\theta}{\frac{2E}{k}(1-\cos^2\theta)^{1/2}} = \theta$$



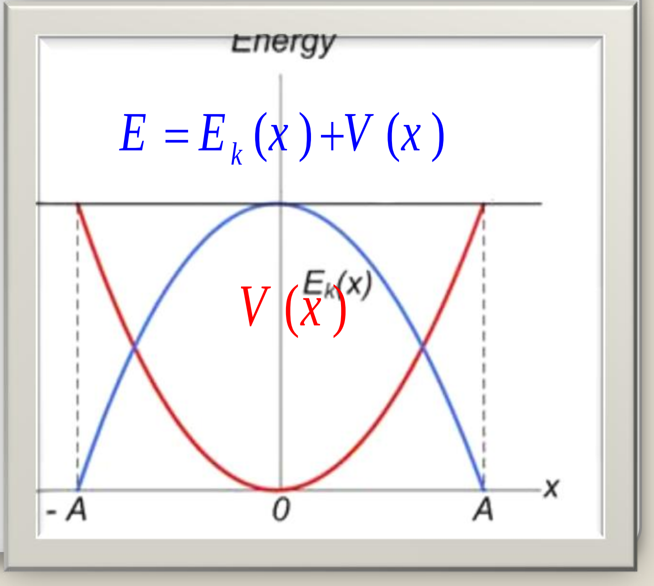
$$\cos^{-1}\frac{x}{\sqrt{2E/k}} = \omega_0 t + C$$



$$x = \sqrt{\frac{2E}{k}}\cos(\omega_0 t + C)$$

$$x(t) = \mathcal{A}\cos(\omega_0 t + \varphi)$$

$$E = \frac{1}{2}mv^2_{max} = \frac{1}{2}k\mathcal{A}^2$$



Example: The energy function of the simple pendulum

$$V = mgh$$

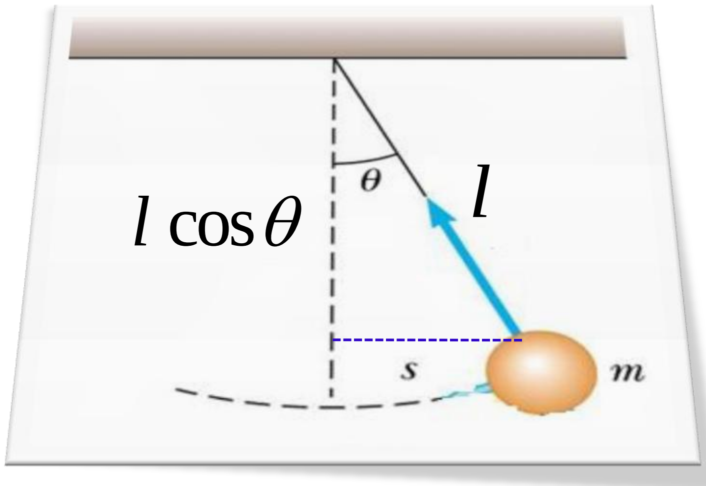
$$h = l - l\cos\theta$$

$$V(\theta) = mgl(1-\cos\theta)$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$$

$$V(\theta) = \frac{1}{2} mgl\theta^2$$

$$V(s) = \frac{1}{2}\frac{mg}{l}s^2$$



$$s = l\theta$$

$$E = \frac{1}{2}m\dot{s}^2 + \frac{1}{2}\frac{mg}{l}s^2$$

Damped Harmonic Motion

$$m\ddot{x} = -kx - c\dot{x}$$

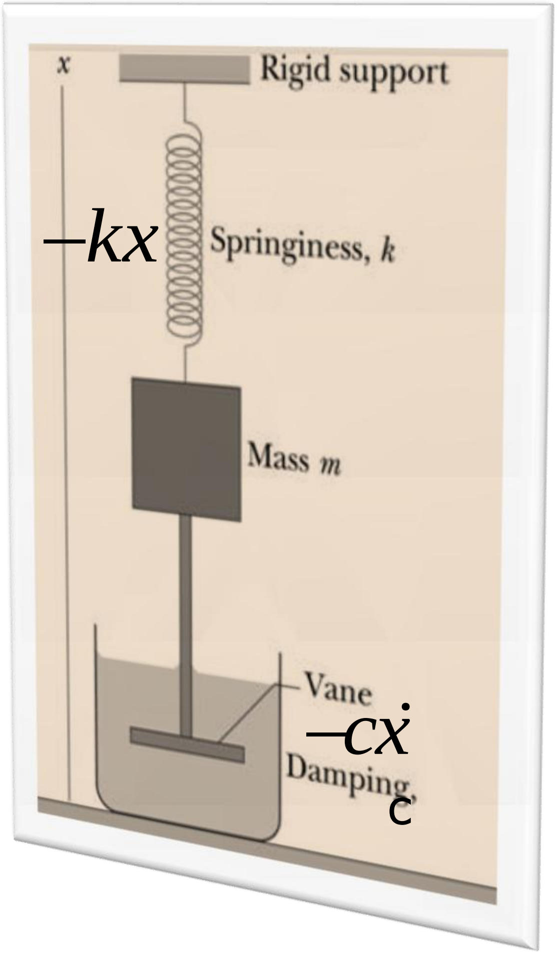
$$m\ddot{x} + c\dot{x} + kx = 0$$

$$x = e^{qt} \quad \dot{x} = qe^{qt}$$

$$\ddot{x} = q^2 e^{qt}$$

$$mq^2 + cq + k = 0$$

$$q = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$



$$\begin{cases} 1)\ c^2 - 4mk > 0 \ \textit{overdamping} \\ 2)\ c^2 - 4mk = 0 \ \textit{critical damping} \\ 3)\ c^2 - 4mk < 0 \ \textit{underdamping} \end{cases}$$

1) $\rightarrow$ *distinct real roots* q_1, q_2

$$q_1 = -\frac{c}{2m} + \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}} = -\gamma_1$$

$$q_2 = -\frac{c}{2m} - \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}} = -\gamma_2$$

$$x(t) = A_1 e^{-\gamma_1 t} + A_2 e^{-\gamma_2 t}$$

$$x(t) = A_1 e^{-\gamma_1 t} + A_2 e^{-\gamma_2 t}$$

$$v(t) = \frac{dx(t)}{dt} = -\gamma_1 A_1 e^{-\gamma_1 t} - \gamma_2 A_2 e^{-\gamma_2 t}$$

$$t = 0, \quad x = x_0, \quad v = 0$$

$$x_0 = A_1 + A_2 \quad 0 = -A_1\gamma_1 - A_2\gamma_2 \quad A_1 = -\frac{\gamma_2}{\gamma_1} A_2$$

$$x_0 = -\frac{\gamma_2}{\gamma_1} A_2 + A_2 \quad x_0 = (\frac{\gamma_1 - \gamma_2}{\gamma_1}) A_2$$

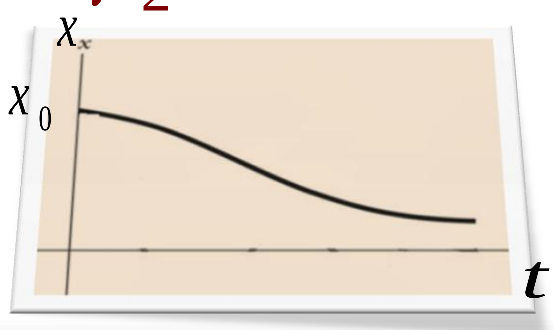
$$A_2 = (\frac{\gamma_1}{\gamma_1 - \gamma_2}) x_0$$

$$A_1 = -\frac{\gamma_2}{\gamma_1 - \gamma_2} x_0 \qquad A_2 = \left(\frac{\gamma_1}{\gamma_1 - \gamma_2}\right) x_0$$

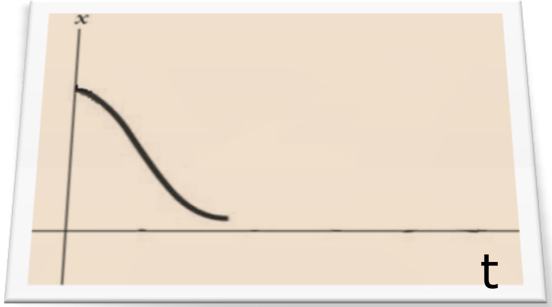
$$x(t) = A_1 e^{-\gamma_1 t} + A_2 e^{-\gamma_2 t}$$

$$x(t) = \frac{-\gamma_2 x_0}{\gamma_1 - \gamma_2} e^{-\gamma_1 t} + \frac{\gamma_1 x_0}{\gamma_1 - \gamma_2} e^{-\gamma_2 t}$$

$$x(t) = \frac{x_0}{\gamma_1 - \gamma_2} \left(-\gamma_2 e^{-\gamma_1 t} + \gamma_1 e^{-\gamma_2 t}\right)$$



2) $c^2 - 4mk = 0$ *critical damping*



equal real roots q

$$q_1 = q_2 = -\frac{c}{2m} = -\gamma$$

$$x(t) = Ae^{-\gamma t} + Bte^{-\gamma t} \qquad A = x_0$$

$$v(t) = \frac{dx}{dt} = -A\gamma e^{-\gamma t} + Be^{-\gamma t} - \gamma Bte^{-\gamma t}$$

$$0 = -\gamma A + B \qquad B = \gamma x_0$$

$$x(t) = x_0 e^{-\gamma t} + x_0 \gamma t e^{-\gamma t} = x_0 e^{-\gamma t}(1 + \gamma t)$$

$$3)\quad c^2 - 4mk < 0 \quad underdamping$$

This is the most interesting case. It occurs if the damping constant *c* is so small that $c^2 < 4Mk$.

$$complex\ \ roots\ q_1, q_2$$

$$q_1 = -\frac{c}{2m} + i\sqrt{\left(\frac{k}{m} - \frac{c^2}{4m^2}\right)} \qquad q_1 = -\frac{c}{2m} - i\sqrt{\left(\frac{k}{m} - \frac{c^2}{4m^2}\right)}$$

$$\gamma = c/2m \qquad \omega_0 = (k/m)^{1/2} \qquad x = e^{qt}$$

$$x_1 = e^{(-\gamma + i\sqrt{\omega_0^2 - \gamma^2})\,t} \qquad x_1 = e^{(-\gamma - i\sqrt{\omega_0^2 - \gamma^2})\,t}$$

$$\omega_d = \sqrt{(\omega_0^{\ 2} - \gamma^2)}$$

$$x_1 = e^{(-\gamma + i\,\omega_d)\,t}$$

$$x_1 = e^{(-\gamma - i\,\omega_d)\,t}$$

$$x(t) = C_+ e^{(-\gamma + i\,\omega_d)\,t} + C_- e^{(-\gamma - i\,\omega_d)\,t}$$

$$x(t) = e^{-\gamma t}\left(C_+ e^{i\,\omega_d t} + C_- e^{-i\,\omega_d t}\right)$$

$$x_1 = e^{-\gamma t}(\cos\omega_d t + i\sin\omega_d t)$$

$$x_2 = e^{-\gamma t}(\cos\omega_d t - i\sin\omega_d t)$$

$$x_3 = \frac{1}{2}(x_1 + x_2) = e^{-\gamma t}\cos\omega_d t$$

$$x_4 = \frac{1}{2i}(x_1 - x_2) = e^{-\gamma t}\sin\omega_d t$$

$$x = e^{-\gamma t}(A\cos\omega_d t + B\sin\omega_d t)$$

$$A = \mathscr{A}\cos\varphi \quad , \quad B = \mathscr{A}\sin\varphi$$

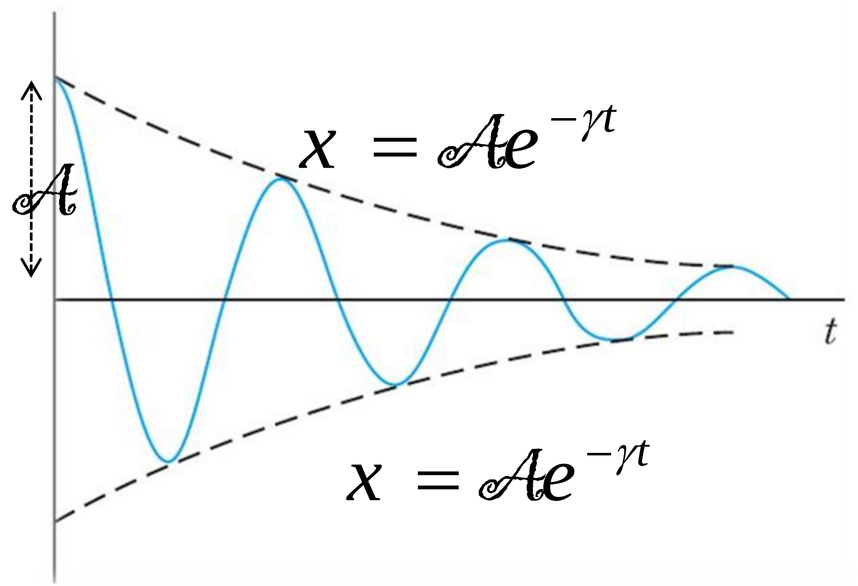
$$x = e^{-\gamma t}\mathscr{A}\cos(\omega_d t - \varphi)$$

$$A^2 + B^2 = \mathcal{A}^2 \sin^2\varphi + \mathcal{A}^2 \cos^2\varphi = \mathcal{A}^2$$

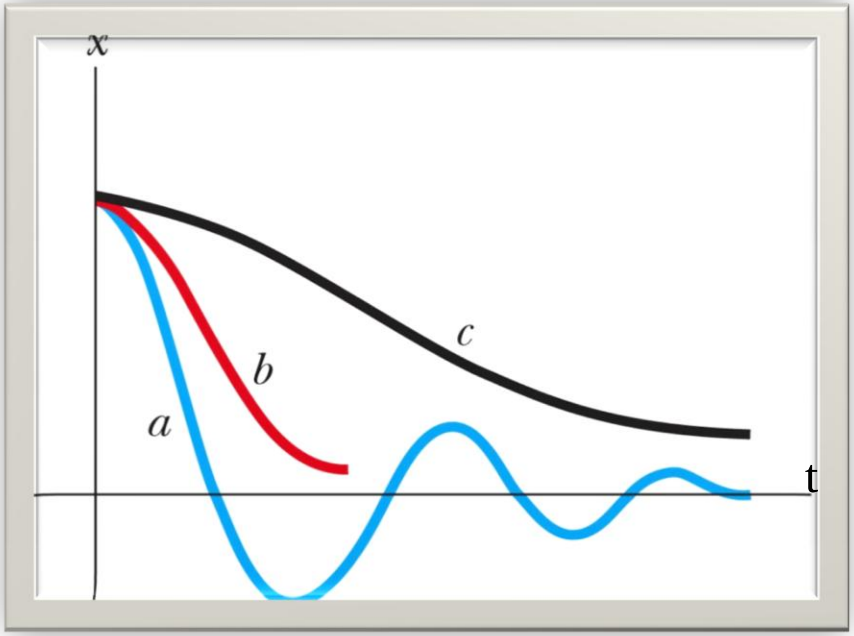
$$B / A = \mathcal{A}\sin\varphi / \mathcal{A}\cos\varphi = \tan\varphi$$

$$C_+ = \frac{1}{2}(A - iB), \quad C_- = \frac{1}{2}(A + iB)$$

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{(\omega_0^2 - \gamma^2)}}$$



Types of Damping



a) an underdamped oscillator
b) a critically damped oscillator
c) an overdamped oscillator

For critically damped and overdamped oscillators there is no periodic motion and the angular frequency ω has a different meaning.