ANALYTICAL MECHANICS 1

Lecture 7

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Chapter 2 Problems

$$1-c)\ F_x = F_0 e^{ct}$$

$$\frac{dv}{dt} = \frac{F(t)}{m} \quad initial\ \ conditions\ \ x(0) = v(0) = 0$$

$$v(t) = \frac{F_0}{cm}(e^{ct} - 1)$$

$$x(t) = \frac{F_0}{c^2 m}(e^{ct} - 1 - ct)$$

$$2-c)\ F_x = F_0 \cos cx$$

$$\text{since } \ddot{x}(t) = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

$$mv\frac{dv}{dx} = F(x) \quad \text{or} \quad v\frac{dv}{dx} = \frac{1}{2}\frac{d(v^2)}{dx}$$

$$\frac{d(v^2)}{dx} = \frac{2F(x)}{m} \qquad v(x) = \sqrt{\frac{2F_0}{cm}\sin cx}$$

$3-c)$ V(x)=?

$$V(x) = -\frac{F_0}{c}\sin cx + const.$$

9) $F(v) = -cv^{3/2}$ $$x_{max} = \frac{2m}{c} v_0^{1/2}$$

$$\ddot{x}(t) = v(x)\frac{dv}{dx}$$ $$mv\frac{dv}{dx} = -cv^{3/2}$$

$$v^{-1/2}dv = -\frac{c}{m}dx$$ $$x(0) = 0 \quad v(0) = v_0$$

$$\int_{v_0}^{v} v^{-1/2}dv = -\frac{c}{m}\int_0^x dx$$ $$2(v^{1/2} - v_0^{1/2}) = -\frac{c}{m}x$$

$$x = \frac{2m}{c}(v_0^{1/2} - v^{1/2})$$

$$v(t)\downarrow, \quad t \to \infty$$

$$x_{max} = \frac{2m}{c} v_0^{1/2}$$

$$10)\quad v^2 = Ae^{-2kx} - \frac{g}{k} \quad upward\ \ motion$$

$$v^2 = \frac{g}{k} - Be^{2kx} \quad downward\ \ motion$$

a) For the upward motion

$$m\dot{v}(t) = -mg - cv^2 \qquad \dot{v} = v\frac{dv}{dx} = \frac{1}{2}\frac{d(v^2)}{dx}$$

$$\frac{dv^2(x)}{dx} = -2g - \frac{2c}{m}v^2(x) = -\frac{2c}{m}\left(v^2(x) + \frac{mg}{c}\right)$$

$$z(x) = v^2(x) + \frac{mg}{c} \qquad \frac{dz}{dx} = -\frac{2c}{m}z \quad k = c/m$$

$$z(x) = v^2(x) + g/k \qquad \frac{dz(x)}{dx} = -2kz(x)$$

$$z(x) = Ae^{-2kx} \qquad v^2(x) = -g/k + Ae^{-2kx}$$

b) For the downward motion

$$m\dot{v}(t) = -mg + cv^2$$

$$\frac{dv^2(x)}{dx} = -2g + \frac{2c}{m}v^2(x) = \frac{2c}{m}\left(v^2 - \frac{mg}{c}\right)$$

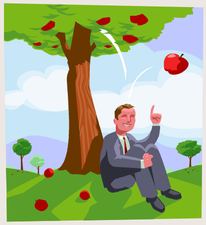
$$z(x) = v^2(x) - \frac{mg}{c} \qquad \frac{dz}{dx} = \frac{2c}{m}z \qquad k = c/m$$

$$z(x) = v^2(x) - g/k$$

$$\frac{dz(x)}{dx} = 2kz(x)$$

$$z(x) = Ae^{2kx}$$

$$v^2(x) = g/k + Ae^{2kx}$$



12) $F(x) = -kx^{-2}$ $\qquad t = \pi\left(\frac{mb^3}{8k}\right)^{1/2}$

$$\ddot{x}(t) = v\frac{dv}{dx} \qquad F = m\ddot{x}$$

$$-kx^{-2} = mv\frac{dv}{dx} \qquad \int_0^v v\,dv = \int_b^x -\frac{kx^{-2}}{m}dx$$

$$\frac{v^2}{2} = \frac{k}{mx}\Big|_b^x \rightarrow \frac{v^2}{2} = \frac{k}{m}\left(\frac{1}{x} - \frac{1}{b}\right)$$

$$\left(\frac{dx}{dt}\right)^2 = \frac{2k}{m}\left(\frac{b-x}{bx}\right) \qquad \int_0^t dt = \int_b^0 \sqrt{\frac{m}{2k}\left(\frac{bx}{b-x}\right)}dx$$

$$t = \sqrt{\frac{mb}{2k}} \int_b^0 \sqrt{\left(\frac{x}{b-x}\right)} dx$$

$$x = b \sin^2 \theta$$

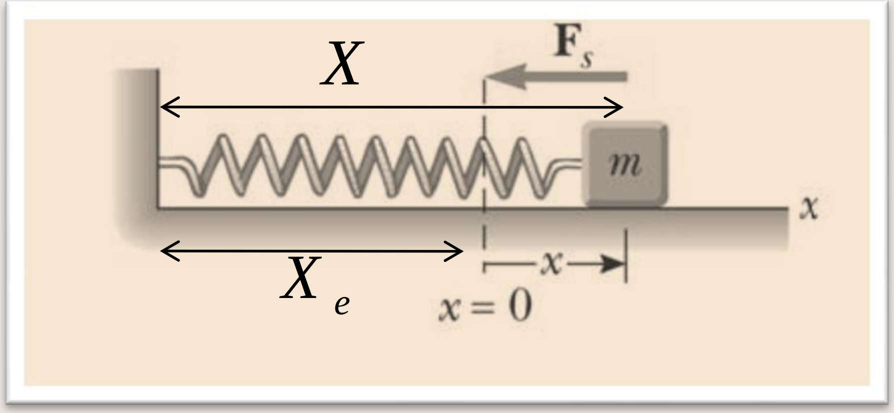
$$t = \pi \left(\frac{mb^3}{8k}\right)^{1/2}$$

Chapter 3

The Harmonic Oscillator

Linear Restoring Force. Harmonic Motion

$$F = -k(X - X_e)$$



$$F = -kx$$

$$-kx = ma$$

$$m\ddot{x} + kx = 0 \qquad x = e^{qt} \qquad \dot{x} = qe^{qt}$$

$$\ddot{x} = q^2 e^{qt} \qquad mq^2 e^{qt} + ke^{qt} = 0$$

$$mq^2 + k = 0 \qquad q^2 = -k/m$$

$$q = \pm i\sqrt{\frac{k}{m}} = \pm i\,\omega_0 \qquad \omega_0 = \sqrt{k/m}$$

$$x(t) = C_+ e^{i\omega_0 t} + C_- e^{-i\omega_0 t} \qquad (1)$$

$$\begin{cases} x_1 = e^{i\omega_0 t} = \cos\omega_0 t + i\sin\omega_0 t \\ x_2 = e^{-i\omega_0 t} = \cos\omega_0 t - i\sin\omega_0 t \end{cases}$$

$$\cos\omega_0 t = \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} \qquad \sin\omega_0 t = \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i}$$

$$\begin{cases} x_3 = \frac{1}{2}x_1 + \frac{1}{2}x_2 = \frac{1}{2}(x_1 + x_2) = \cos\omega_0 t \\ x_4 = \frac{1}{2i}x_1 - \frac{1}{2i}x_2 = \frac{1}{2i}(x_1 - x_2) = \sin\omega_0 t \end{cases}$$

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t \qquad (2)$$

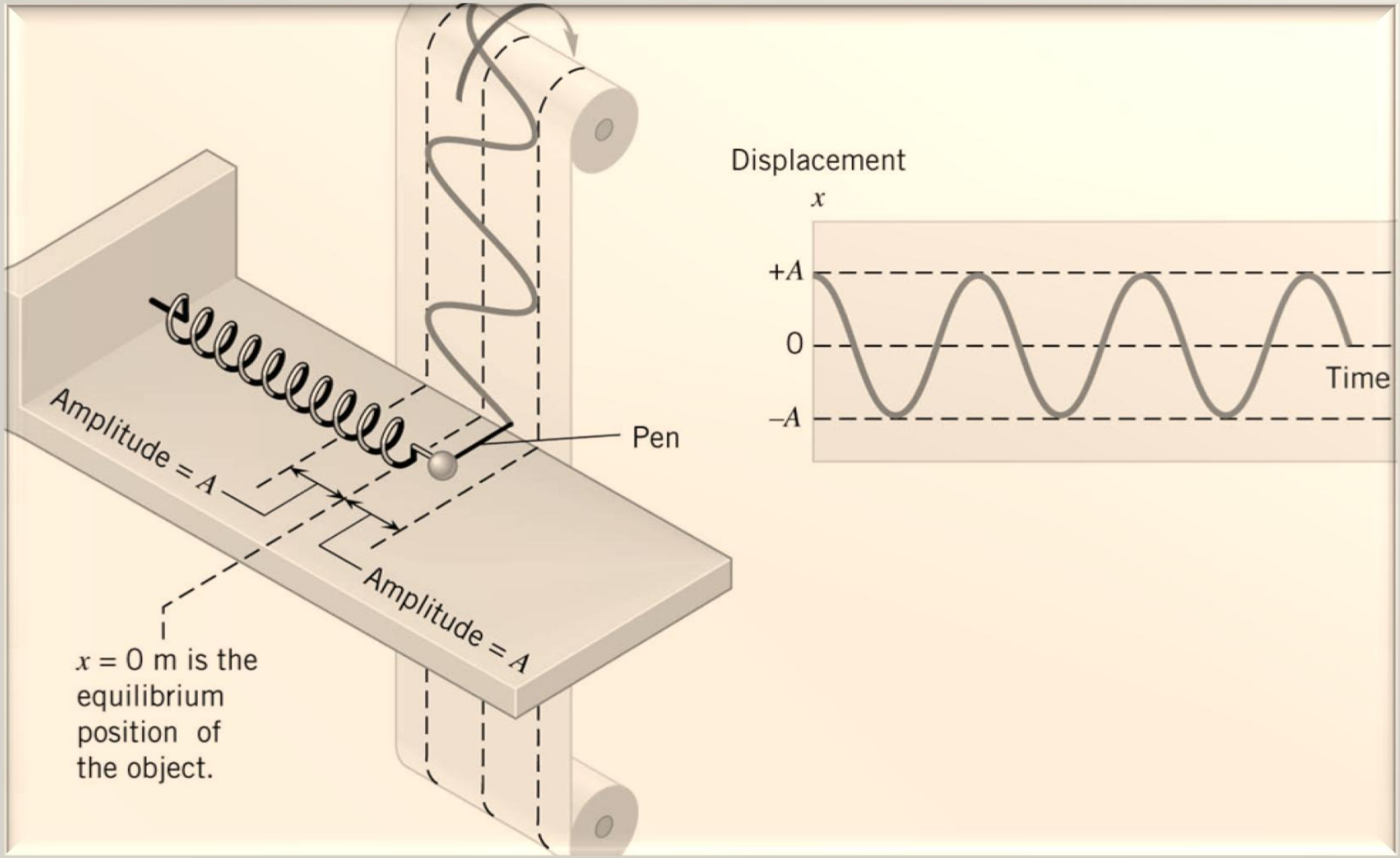
$$C_+ = \frac{1}{2}(A - iB), \quad C_- = \frac{1}{2}(A + iB)$$

$$A = \mathcal{A}\cos\varphi \quad , \quad B = \mathcal{A}\sin\varphi$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$x(t) = \mathcal{A}\cos(\omega_0 t - \varphi) \qquad (3)$$

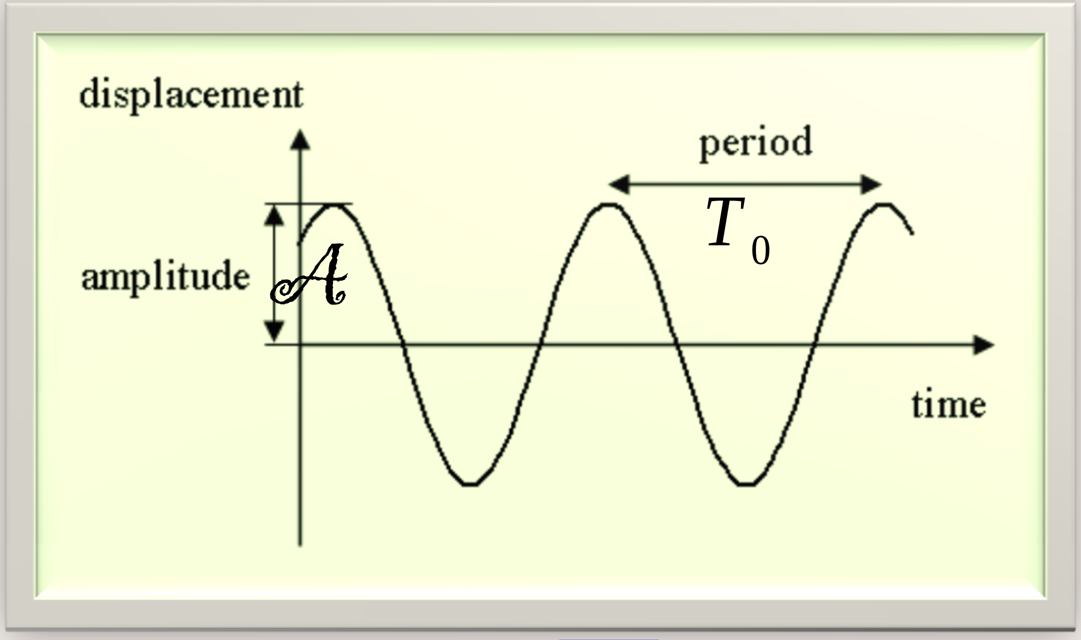
Simple Harmonic Motion



$$x(t) = \mathcal{A}\cos(\omega_0 t - \varphi)$$

$$A^2 + B^2 = \mathcal{A}^2 \sin^2\varphi + \mathcal{A}^2\cos^2\varphi = \mathcal{A}^2$$

$$B/A = \mathcal{A}\sin\varphi / \mathcal{A}\cos\varphi = \tan\varphi$$



$$T_0\omega_0 = 2\pi$$

$$\omega_0 = 2\pi f_0$$

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$$

$$f_0 = \frac{1}{T_0} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Constants of the Motion and Initial Conditions

$$t = 0, \quad x = x_0, \quad \dot{x} = \dot{x}_0 \qquad x_0 = A$$

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

$$\dot{x}(t) = -A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t$$

$$\dot{x}_0 = B\omega_0$$

$$x(t) = x_0\cos\omega_0 t + \frac{\dot{x}_0}{\omega_0}\sin\omega_0 t$$

$$\mathcal{A} = (A^2 + B^2)^{1/2} = (x_0^2 + \frac{\dot{x}_0^2}{\omega_0^2})^{1/2}$$

$$C_+ = \frac{1}{2}(A - iB), \quad C_- = \frac{1}{2}(A + iB)$$

$$C_+ = \frac{1}{2}(x_0 - i\frac{\dot{x}_0}{\omega_0}), \quad C_- = \frac{1}{2}(x_0 + i\frac{\dot{x}_0}{\omega_0})$$

$$\tan\varphi = B/A \quad \rightarrow \varphi = \tan^{-1}(\frac{\dot{x}_0}{x_0\omega_0})$$

Effect of a Constant External Force on Harmonic Oscillator

$$F = -k(X - X_e) + mg$$

$$F = -kx + mg$$

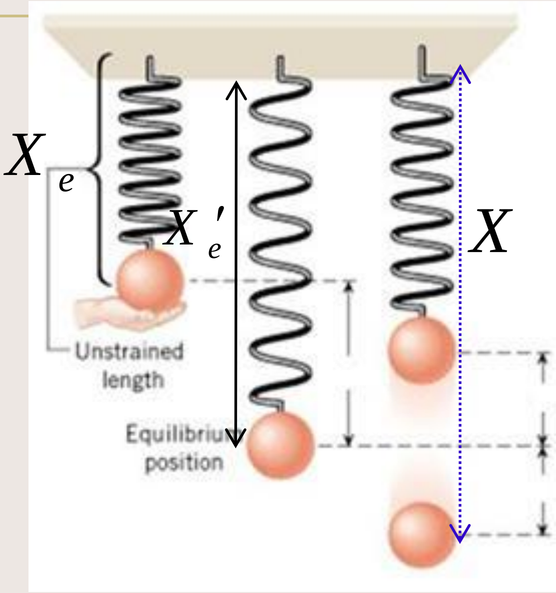
$$0 = -k(X_e' - X_e) + mg$$

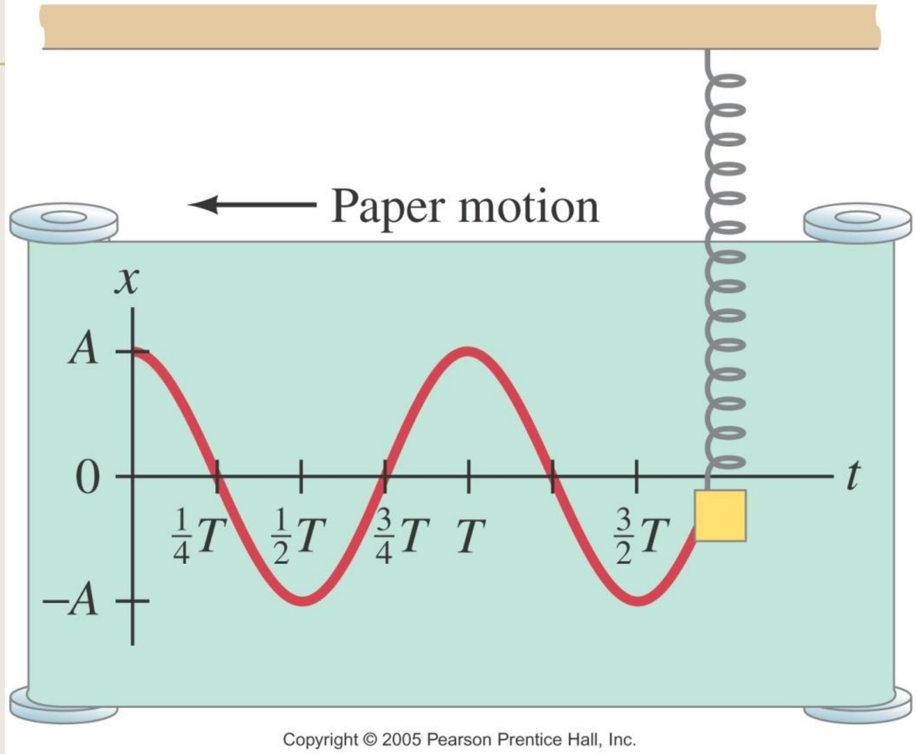
$$X_e' = X_e + mg/k$$

$$x = X - X_e' = X - X_e - mg/k$$

$$F = -kx + mg \qquad F = -kx$$

$$m\ddot{x} + kx = 0$$




Paper motion
x
A
0
−A
t
¼T
½T
¾T
T
3/2T
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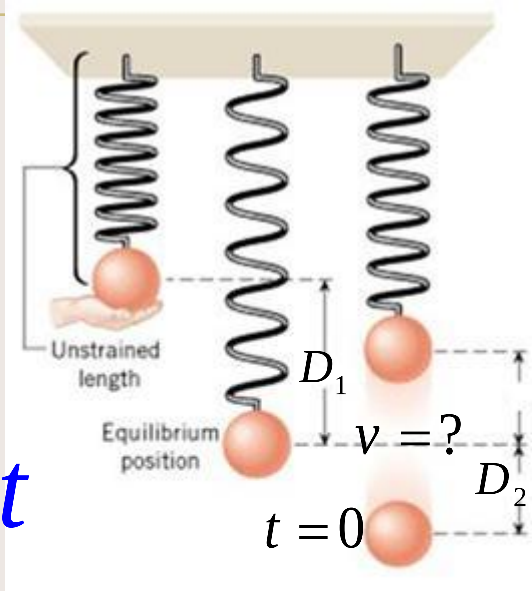
Example (1): $F_x = 0 = -kD_1 + mg$

$$k = \frac{mg}{D_1} \quad \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{D_1}}$$



$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

$$\dot{x}(t) = -A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t$$

$x_0 = D_2 = A$ $\quad \dot{x}_0 = 0 = B\omega_0$ $\quad B = 0$

$$x(t) = D_2 \cos\left(\sqrt{\frac{g}{D_1}}t\right)$$

$$\dot{x}(t) = -D_2\sqrt{\frac{g}{D_1}}\sin\left(\sqrt{\frac{g}{D_1}}t\right) \qquad \omega_0 t = \frac{2\pi}{T_0}\cdot\frac{T_0}{4} = \frac{\pi}{2}$$

$$\ddot{x}(t) = -D_2\frac{g}{D_1}\cos\left(\sqrt{\frac{g}{D_1}}t\right) \qquad \cos\pi = -1$$

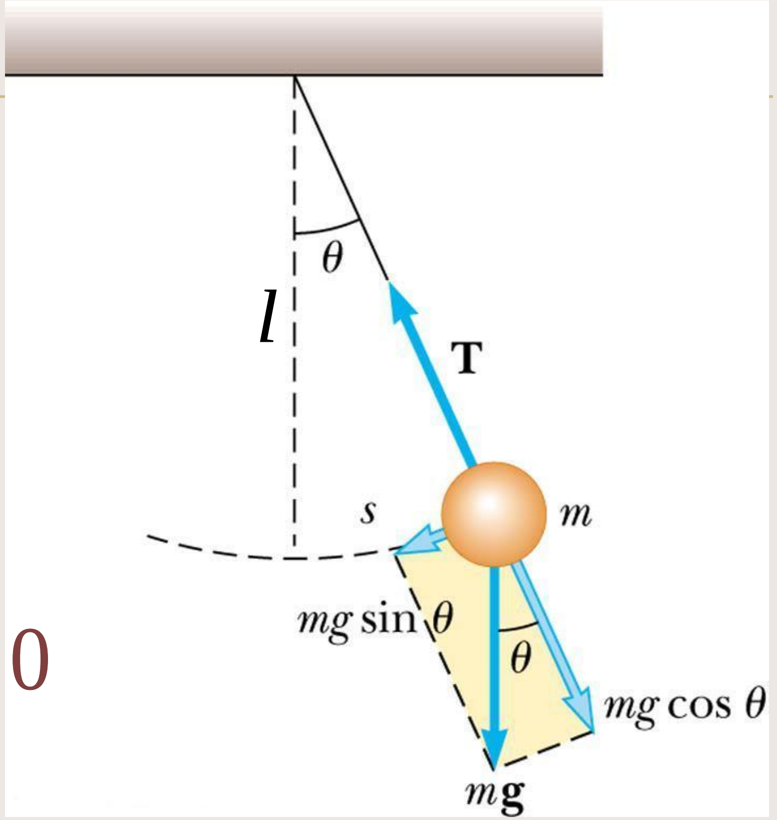
$$\dot{x} = -D_2\sqrt{\frac{g}{D_1}} \quad (center) \qquad \ddot{x} = D_2\frac{g}{D_1} \quad (top)$$

$$D_1 = D_2 \rightarrow \ddot{x} = g$$

Example (2): The simple pendulum

$$m\ddot{s} = -mg\sin\theta$$

$$s = l\theta, \quad \sin\theta = \theta$$



$$ml\ddot{\theta} = -mg\theta$$

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \qquad \ddot{x} + \frac{k}{m}x = 0$$

$$\ddot{s} + \frac{g}{l}s = 0 \qquad \omega_0 = \sqrt{g/l} \qquad T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}}$$