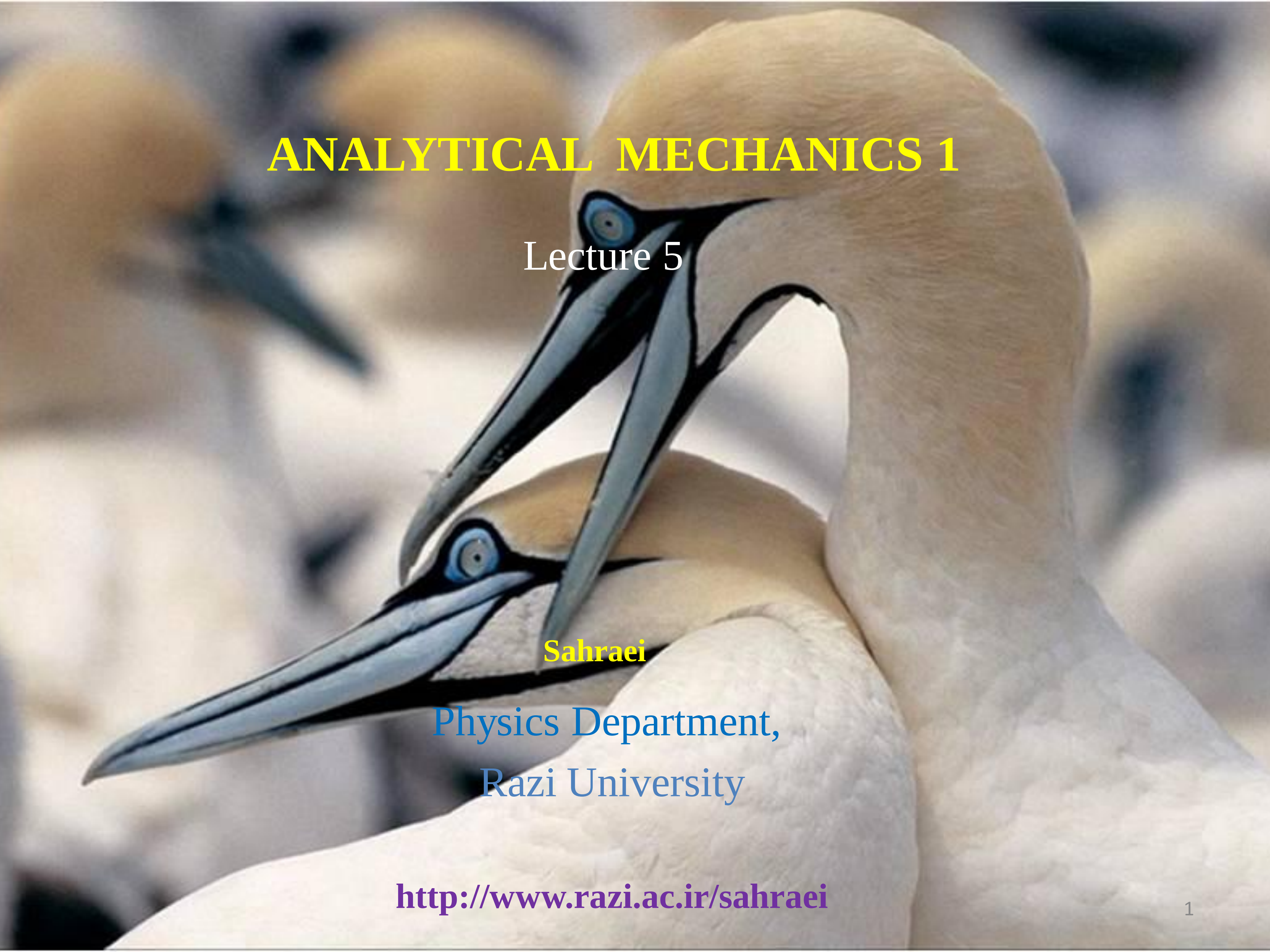

ANALYTICAL MECHANICS 1

Lecture 5

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Forces That Depend on Position
The Concepts of Kinetic and Potential Energy

$$F_x(x,\dot{x},t) = m\ddot{x} \qquad F(x) = m\ddot{x}$$

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{dx}{dt}\frac{d\dot{x}}{dx} = v\frac{dv}{dx}$$

$$F(x) = mv\frac{dv}{dx} = \frac{m}{2}\frac{d(v^2)}{dx} = \frac{dT}{dx}$$

$$\boxed{T = \frac{1}{2}mv^2}$$

$$\int_{x_0}^{x} F(x)dx = T - T_0 \qquad V(x) = \int_{x}^{x_0} F(x)dx = -\int_{x_0}^{x} F(x)dx$$

$$-\frac{dV}{dx} = F(x)$$

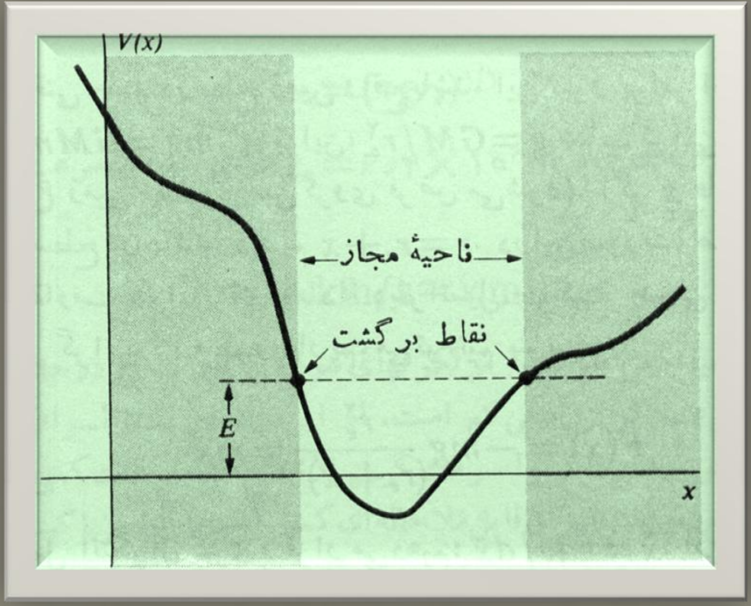
$$\int_{x_0}^{x} F(x)dx = -\int_{x_0}^{x} dV = -V(x) + V(x_0) = T - T_0$$

$$-[V(x) + C] + [V(x_0) + C] = -V(x) + V(x_0)$$

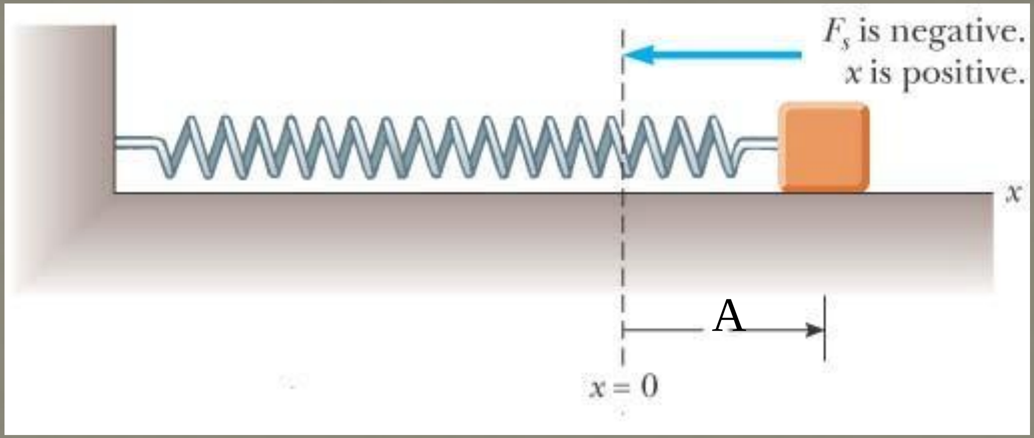
$$T + V(x) = T_0 + V(x_0) = const. = E$$

$$\frac{1}{2}mv^2 + V(x) = E \quad v = \frac{dx}{dt} = \pm\sqrt{\frac{2}{m}[E - V(x)]}$$

$$\sqrt{\frac{m}{2}}\int_{x_0}^{x} [E - V(x)]^{-1/2} dx = t - t_0 \qquad V(x) \le E$$


V(x)
ناحیهٔ مجاز
نقاط برگشت
E
x

Example 1: Spring



$$F = -kx$$

$$-\frac{dV}{dx} = F(x)$$

$$V(x) = -\int_0^x -kx\,dx = \frac{1}{2}kx^2$$

$$\sqrt{\frac{m}{2}}\int_{x_0}^{x}\left(E - \frac{1}{2}kx^2\right)^{-1/2}dx = t$$

$$\sqrt{\frac{m}{2E}}\int_{x_0}^{x}\left(1 - \frac{1}{2E}kx^2\right)^{-1/2}dx = t$$

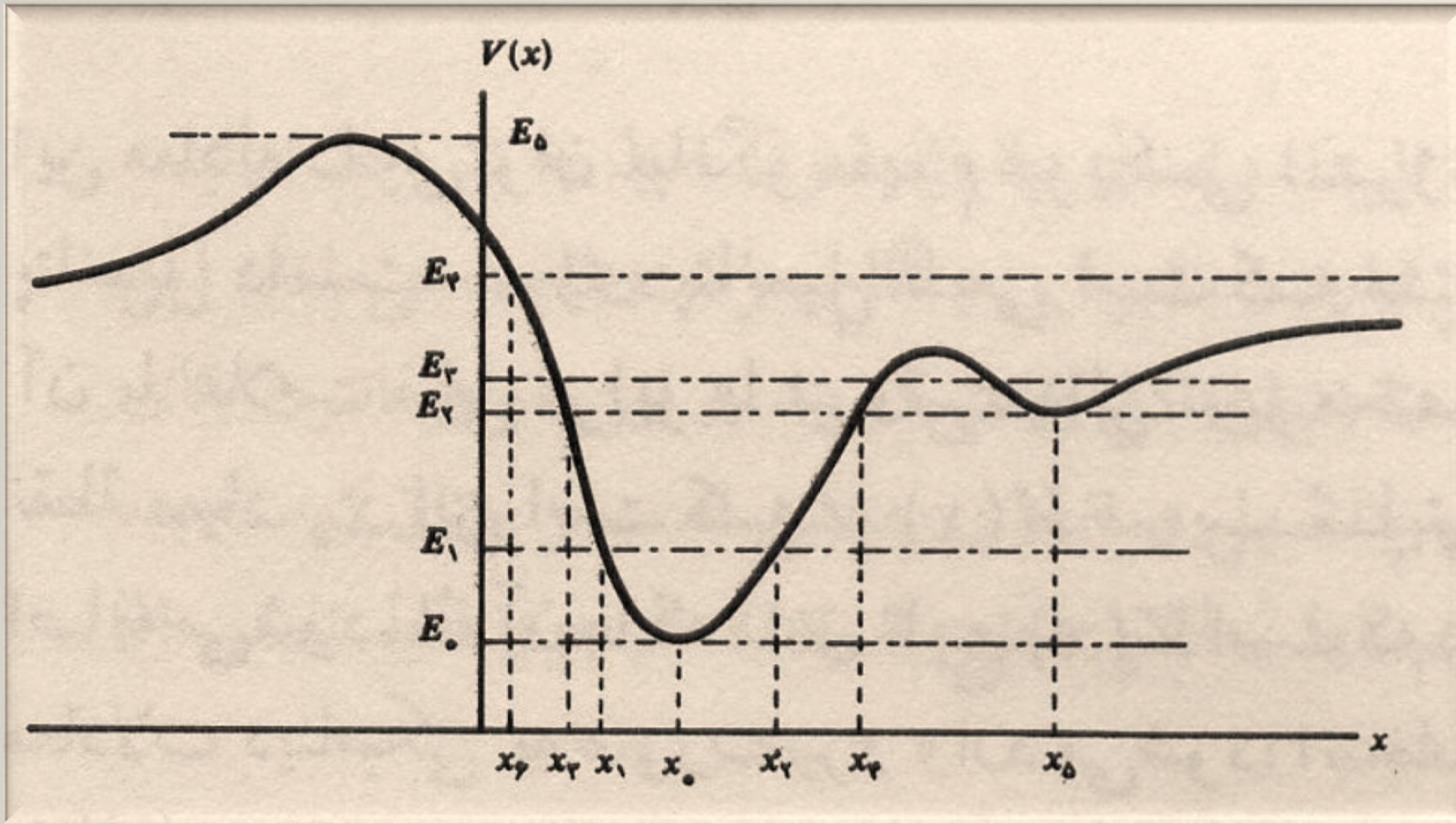
$$\sqrt{\frac{k}{2E}}x = \sin\theta \qquad \sqrt{\frac{k}{2E}}dx = \cos\theta d\theta$$

$$t = \sqrt{\frac{m}{2E}}\sqrt{\frac{2E}{k}}\int_{\theta_0}^{\theta}\frac{\cos\theta d\theta}{\cos\theta} = \sqrt{\frac{m}{k}}(\theta - \theta_0)$$

$$\omega = \sqrt{\frac{k}{m}} \qquad \theta = \omega t + \theta_0$$

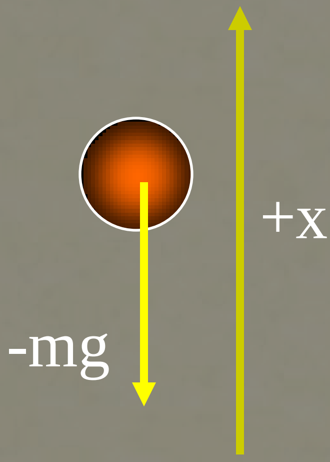
$$x = \sqrt{\frac{2E}{k}}\sin\theta \qquad x = A\sin(\omega t + \theta_0)$$

$$A = \sqrt{\frac{2E}{k}} \qquad E = \frac{1}{2}kA^2$$


V(x)
E۵
E۴
E۳
E۲
E۱
E۰
x
x۴ x۳ x۱ x۰ x'۱ x۲ x۵

Example 2: Free fall

$$-\frac{dV}{dx} = -mg \qquad V = mgx + C$$



$$\frac{1}{2}mv^2 + mgx = E$$

$$E = \frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + mgx$$

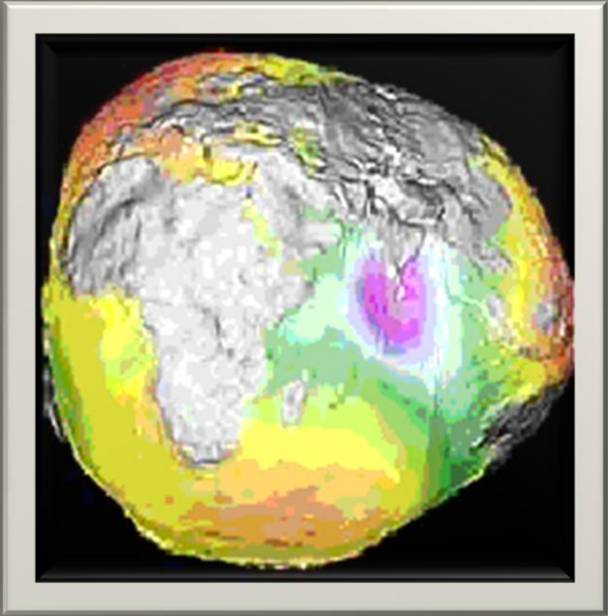
$$v^2 = v_0^2 - 2gx \qquad 0 = v_0^2 - 2gx_{\max}$$

$$h = x_{\max} = \frac{v_0^2}{2g}$$

Example 3: Variation of gravity with height

$$\vec{g}^{\,*} = \vec{g}$$

در صورت عدم چرخش زمین

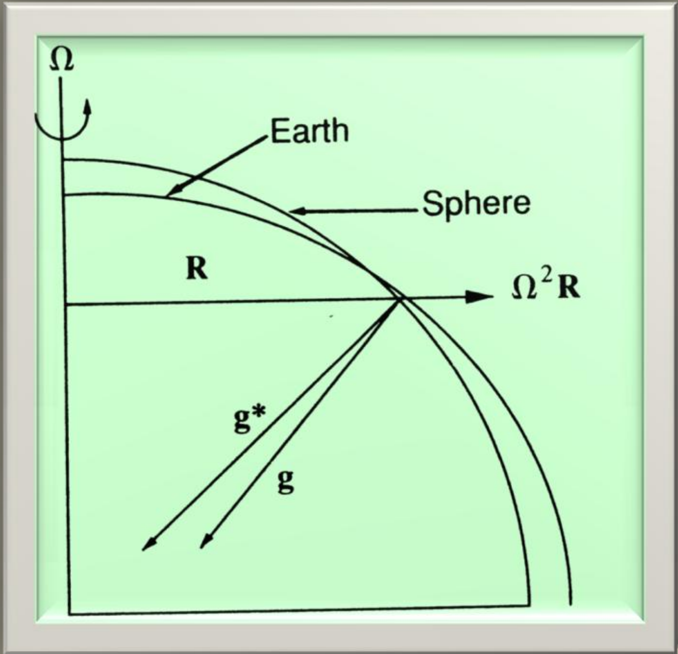


$$\vec{g}^{\,*} = \vec{g} \rightarrow pole$$

$$g = g^{*} - \Omega^2 R \rightarrow Equator$$

$$\Delta g = g_{pol} - g_{Eq} = 5.2 cm/s^2$$

$$g_x = 0, g_y = 0, g_z = -g$$



$$F_r = -\frac{GMm}{r^2} \qquad -mg = -GMm / r_e^{\,2}$$

$$g = GM / r_e^{\,2} \qquad F(x) = -mg\frac{r_e^{\,2}}{(r_e + x)^2} = m\ddot{x}$$

$$\ddot{x} = v\frac{dv}{dx} \qquad -mgr_e^{\,2}\int_{x_0}^{x}\frac{dx}{(r_e + x)^2} = \int_{v_0}^{v} mv\,dv$$

$$mgr_e^{\,2}\left(\frac{1}{r_e + x} - \frac{1}{r_e + x_0}\right) = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^{\,2}$$

$$\int_{x_0}^{x} F(x)dx = -\int_{x_0}^{x} dV = -V(x) + V(x_0) = T - T_0$$

$$V(x) = -mg\left[r_e^2/(r_e + x)\right]$$

$$2gr_e^2\left(\frac{-x}{(r_e + x)r_e}\right) = v^2 - v_0^2$$

$$-2gx\left(\frac{1}{1 + \frac{x}{r_e}}\right) = v^2 - v_0^2$$

$$v^2 = v_0^2 - 2gx\left(1 + \frac{x}{r_e}\right)^{-1}$$

$$if \quad x \ll r_e \qquad v^2 = v_0^2 - 2gx$$

$$if \quad v = 0 \rightarrow v_0^2 = 2gx_{max}\left(1 + \frac{x_{max}}{r_e}\right)^{-1}$$

$$x_{max} = h = \frac{v_0^2}{2g}\left(1 - \frac{v_0^2}{2gr_e}\right)^{-1}$$

$$if \quad v_0^2 \ll 2gr_e \rightarrow x_{max} = h = \frac{v_0^2}{2g}$$

$$x_{\max} = \infty \rightarrow 1 - \frac{v_0^2}{2gr_e} = 0$$

$$v_e = (2gr_e)^{1/2} \qquad v_e \simeq 11 km/s$$

$$for\ \mathrm{O}_2\ \&\ N_2 \qquad v_e \simeq 0.5 km/s$$

The force as a Function of Time. The concept of Impulse

$$F(t) = m\frac{dv}{dt} \qquad \int_0^t F(t)dt = mv(t) - mv_0$$

$$\frac{dx}{dt} = v(t) = v_0 + \int_o^t \frac{F(t)}{m}dt$$

$$x - x_0 = \int_0^t v(t)dt = v_0 t + \int_0^t \left[\int_0^t \frac{F(t')}{m} dt' \right] dt$$

Example 1: Constant force

$$v(t) = v_0 + \frac{F}{m}\int_0^t dt = v_0 + \frac{Ft}{m}$$

$$x(t) = x_0 + v_0 t + \int_0^t \frac{Ft}{m} dt = x_0 + v_0 t + \frac{Ft^2}{2m}$$

$$\dot{x} = v = at + v_0 \qquad x = \frac{1}{2}at^2 + v_0 t + x_0$$

Example 2: Step force

$$x(t) = x_0 + v_0 t + \left(\frac{F_1 t^2}{2m}\right) \qquad v = v_0 + \frac{F_1 t}{m}$$

$$v(t) = v_0 + \frac{F_1 t_1}{m} + \frac{F_2}{m}(t - t_1)$$

$$x(t) = x_0 + v_0 t + \frac{F_1 t_1^2}{2m} + \left(v_0 + \frac{F_1 t_1}{m}\right)(t - t_1) + \frac{F_2}{2m}(t - t_1)^2$$

Example 3: Uniformly increasing force

$$t = 0 \qquad F(t) = ct$$

$$m\frac{dv}{dt} = ct \qquad v = \int_0^t \frac{ct}{m}dt = \frac{ct^2}{2m}$$

$$x = \int_0^t \frac{ct^2}{2m}dt = \frac{ct^3}{6m}$$

$$\frac{d^3x}{dt^3} = \frac{c}{m}$$

The time-rate of change of acceleration (jerk)

Velocity-Dependent Forces
Fluid Resistance and Terminal Velocity

$$F_0 + F(v) = m\frac{dv}{dt}$$

$$F_0 + F(v) = mv\frac{dv}{dx}$$

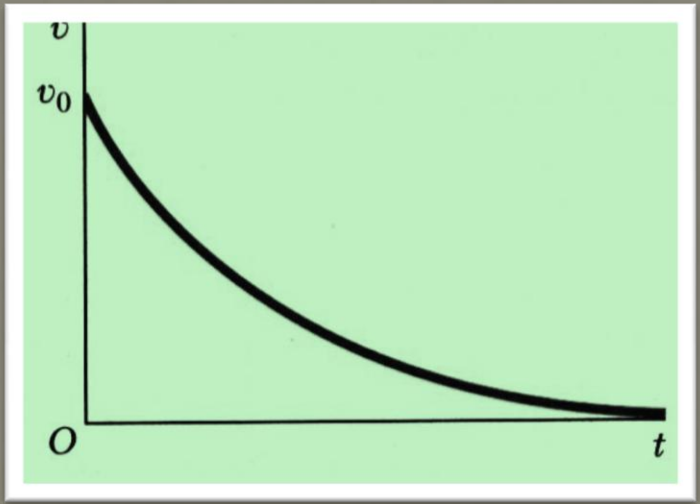
$$F(v) = -c_1 v\,, \quad F_0 = 0$$

Example 1: Horizontal motion with linear resistance

$$-c_1 v = m\frac{dv}{dt} \qquad dt = -\frac{m\,dv}{c_1 v}$$

$$t = \int_{v_0}^{v} -\frac{m\,dv}{c_1 v} = -\frac{m}{c_1}\ln\left(\frac{v}{v_0}\right)$$

$$v = v_0 e^{-c_1 t/m}$$

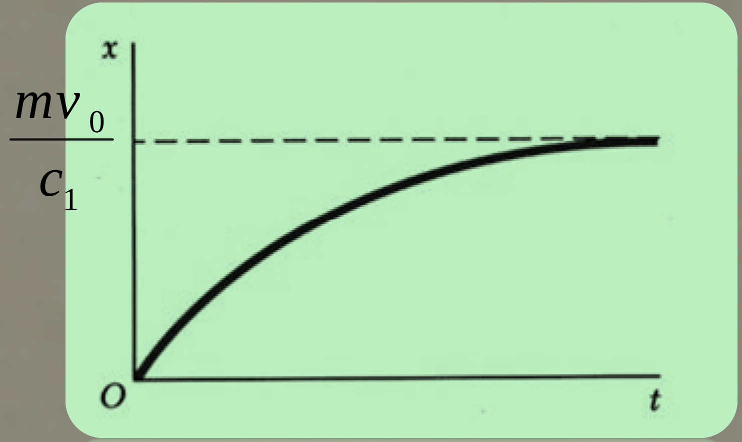


$$\frac{dx}{dt} = v_0 e^{-c_1 t/m} \qquad x = \int_0^t v_0 e^{-c_1 t/m} dt$$

$$x = -\frac{mv_0}{c_1} e^{-c_1 t/m} + \frac{mv_0}{c_1} = \frac{mv_0}{c_1}(1 - e^{-c_1 t/m})$$

$$if \quad t \to \infty \ , \ e^{-c_1 t/m} = 0$$

$$\Rightarrow x_{\lim} \to \frac{mv_0}{c_1}$$

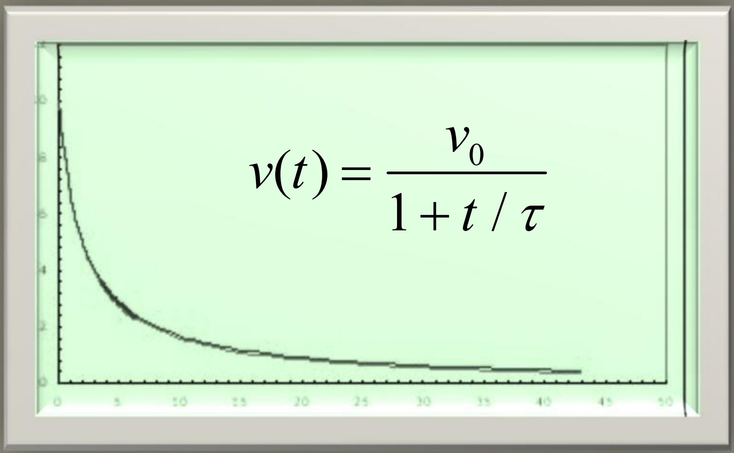


Example 2: Horizontal motion with quadratic resistance

$$-c_2 v^2 = m\frac{dv}{dt} \qquad \int_0^t dt = -\int_{v_0}^{v} \frac{m}{c_2}\frac{dv}{v^2}$$

$$t = \frac{m}{c_2}\left(\frac{1}{v} - \frac{1}{v_0}\right) \qquad \frac{1}{v} = \frac{c_2}{m}t + \frac{1}{v_0}$$

$$v = \frac{v_0}{1 + \frac{c_2 v_0}{m}t} = \frac{v_0}{1+kt}$$



$$\frac{dx}{dt} = \frac{v_0}{1+kt} \qquad \int_0^x dx = \int_0^t \frac{v_0}{1+kt} dt$$

$$x = \frac{v_0}{k} \ln(1+kt)\Big|_0^t = \frac{m}{c_2} \ln\left(1+\frac{v_0 c_2}{m} t\right)$$