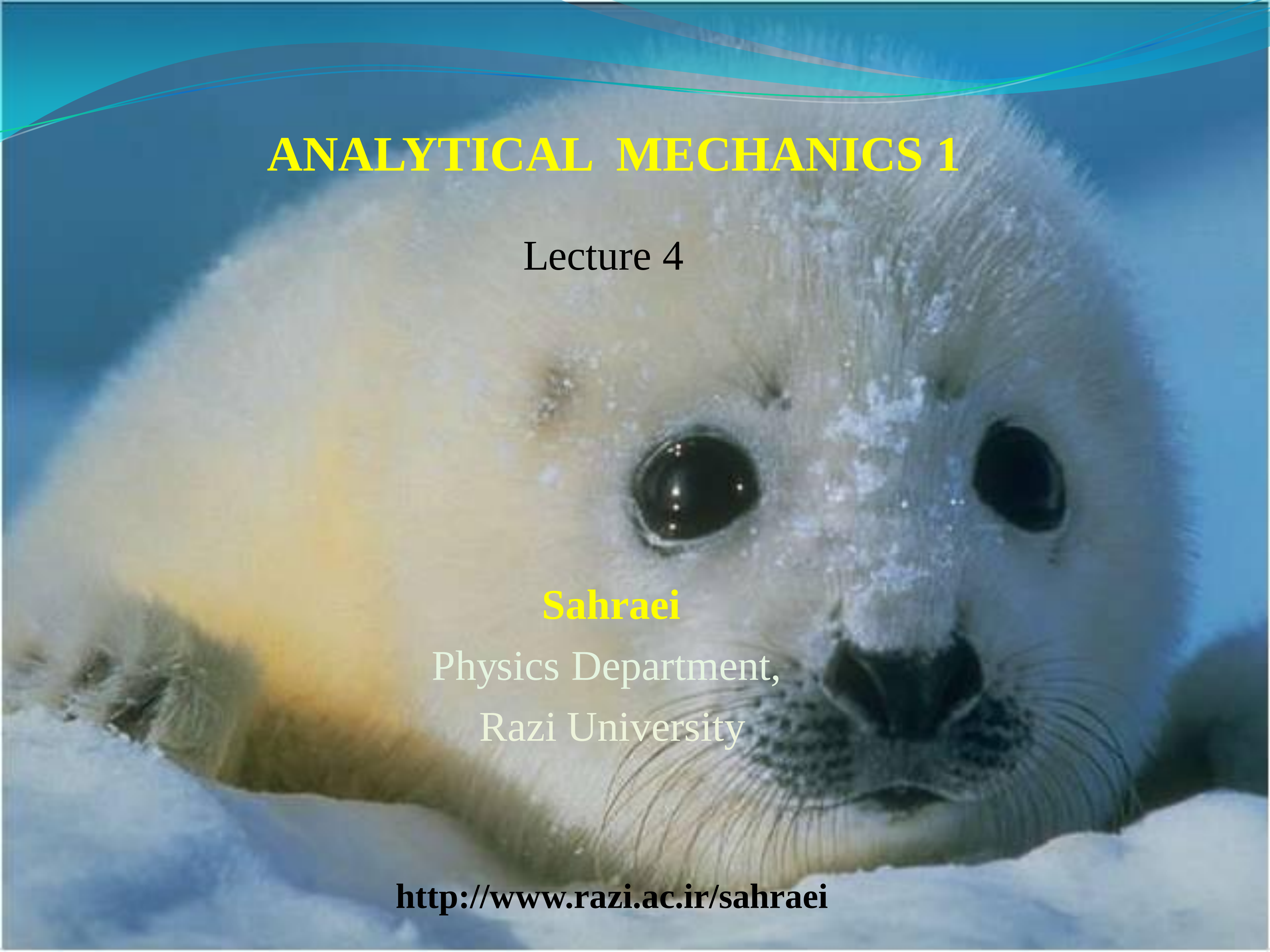


ANALYTICAL MECHANICS 1

Lecture 4

Sahraei

Physics Department,

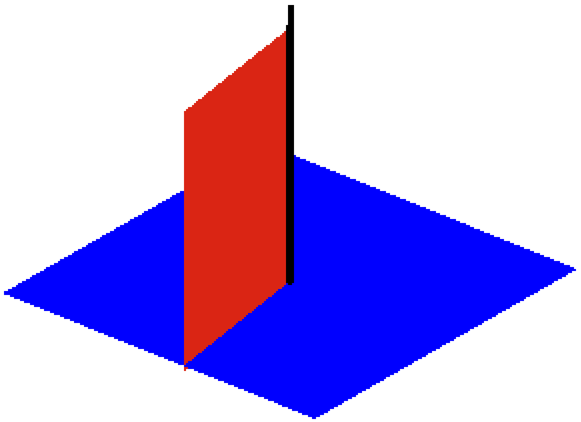
Razi University

http://www.razi.ac.ir/sahraei

Cylindrical Coordinate Surfaces

The three orthogonal components, *R*, *Θ*, and *z* and increasing at a constant rate.

Green: Increasing *R*.
Blue: Increasing *z*.
Red: Increasing *Θ*



$$0 \le R \le \infty$$

Position vector, $\vec{r} = R\,\hat{e}_R + z\,\hat{e}_z$

$$0 \le \theta < 2\pi$$

$$-\infty < z < \infty$$

Example(1): $R = b, \quad \theta = \omega t, \quad z = ct$

$$\dot{R} = \ddot{R} = 0, \quad \dot{\theta} = \omega, \quad \ddot{\theta} = 0, \quad \dot{z} = c, \quad \ddot{z} = 0$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{R}\,\hat{e}_R + R\dot{\theta}\hat{e}_\theta + \dot{z}\hat{e}_z$$ Velocity vector

$$\vec{v} = b\omega\hat{e}_\theta + c\hat{e}_z$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \left(\ddot{R} - R\dot{\theta}^2\right)\hat{e}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\hat{e}_\theta + \ddot{z}\,\hat{e}_z$$ Acceleration vector

$$\vec{a} = \frac{d\vec{v}}{dt} = -b\omega^2\,\hat{e}_R$$

Velocity & Acceleration in Spherical Coordinate

In spherical coordinate systems the **unit vectors are not constant**. Their direction changes when the position of the point P changes

$\hat{e}_r$: parallel to **r**

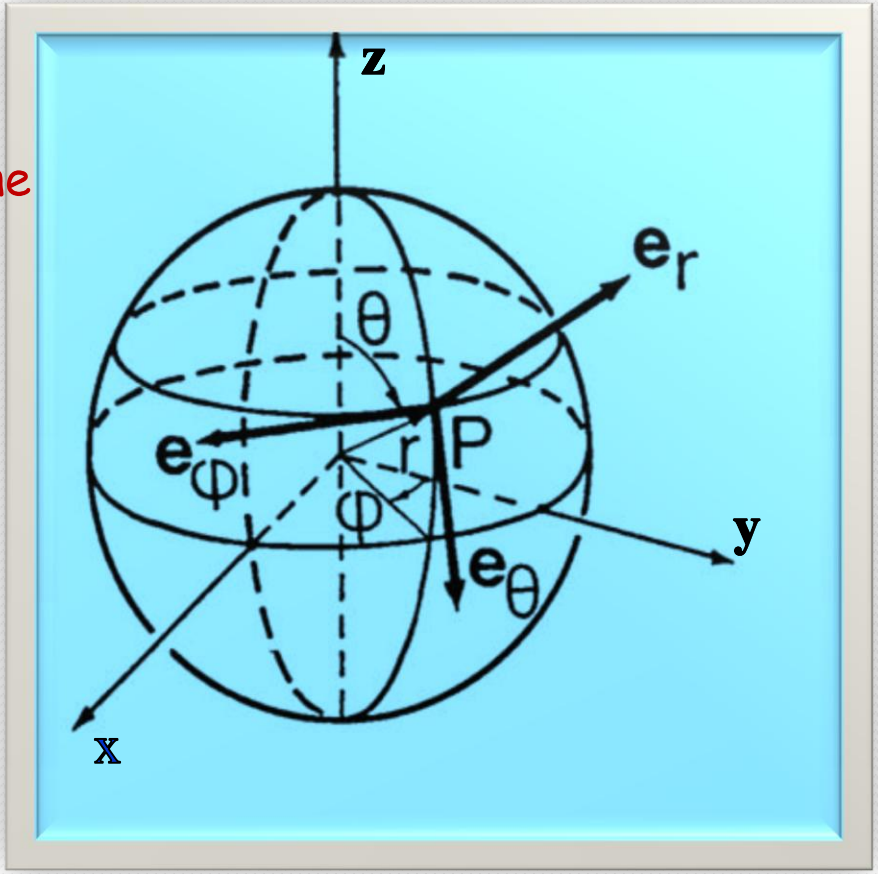
$\hat{e}_\theta$: perpendicular to **r** in the *r-z* plane

$\hat{e}_\varphi$: perpendicular to the *r-z* plane

$$\hat{e}_r \times \hat{e}_\theta = \hat{e}_\varphi$$

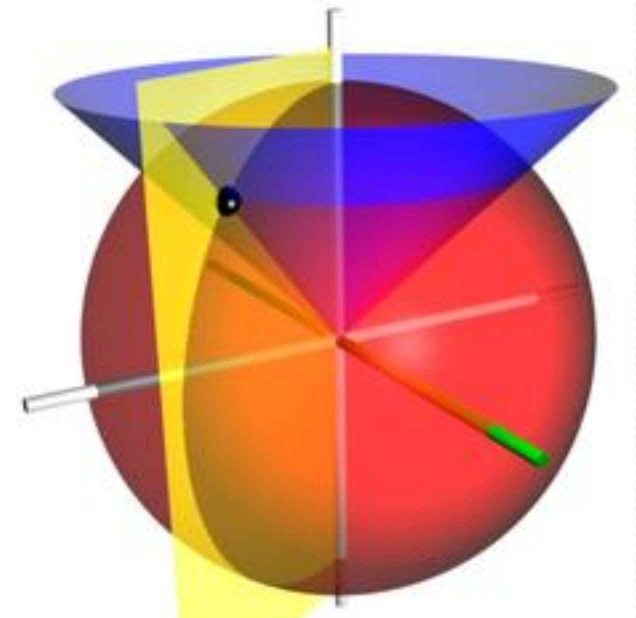
$$\hat{e}_\theta \times \hat{e}_\varphi = \hat{e}_r$$

$$\hat{e}_\varphi \times \hat{e}_r = \hat{e}_\theta$$



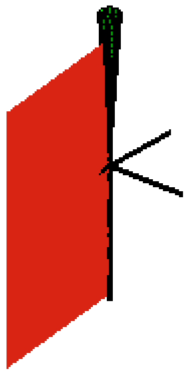
$r \geq 0 \quad 0 \leq \theta \leq \pi \quad 0 \leq \varphi < 2\pi$

Illustration of spherical coordinates



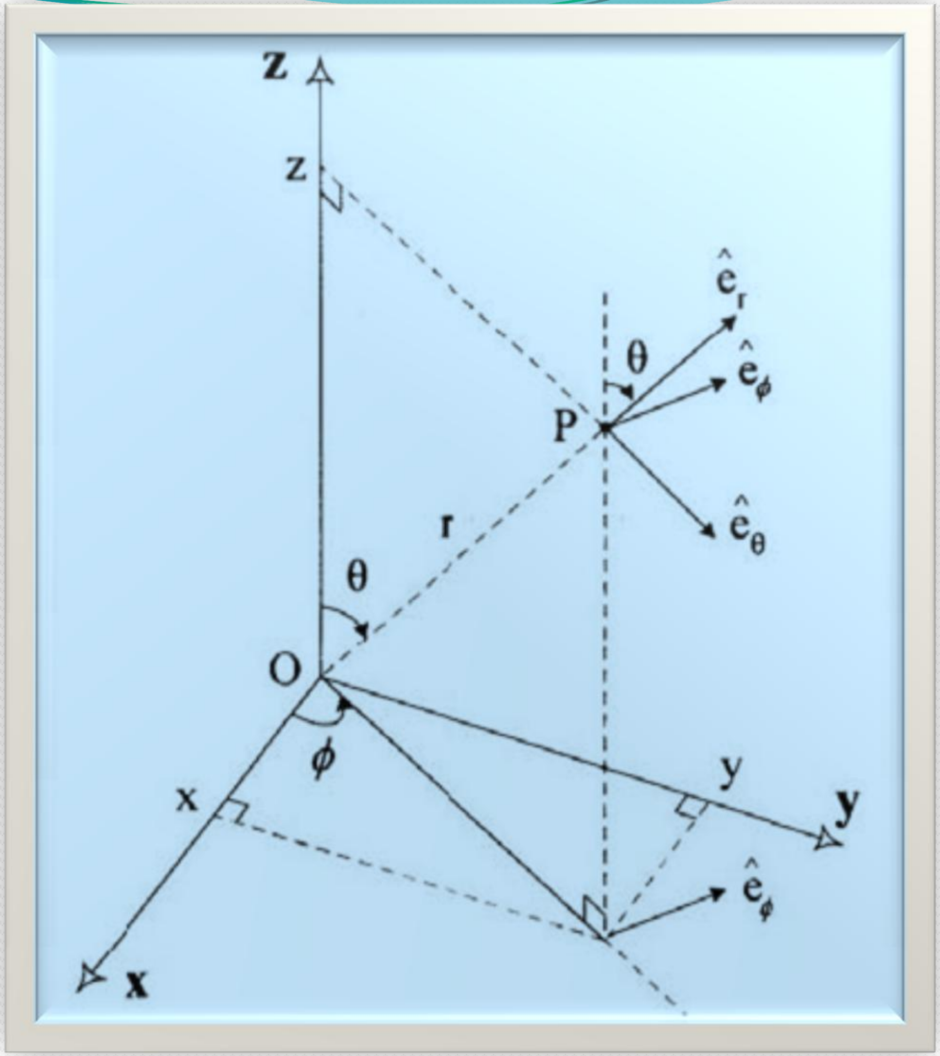
The red sphere shows the points with $r = 2$,
the blue cone shows the points with inclination (or elevation) $\theta = 45°$,

the yellow half-plane shows the points with azimuth $\varphi = -60°$.
The zenith direction is vertical, and the zero-azimuth axis is highlighted in green.



The spherical coordinates (2,45°,−60°) determine the point of space where those three surfaces intersect, shown as a black sphere

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$



$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arctan\left(\sqrt{x^2 + y^2}/z\right) \\ \varphi = \arctan(y/x) \end{cases}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = r\sin\theta\cos\varphi\,\hat{\mathrm{i}} + r\sin\theta\sin\varphi\,\hat{\mathrm{j}} + r\cos\theta\,\hat{\mathrm{k}}$$

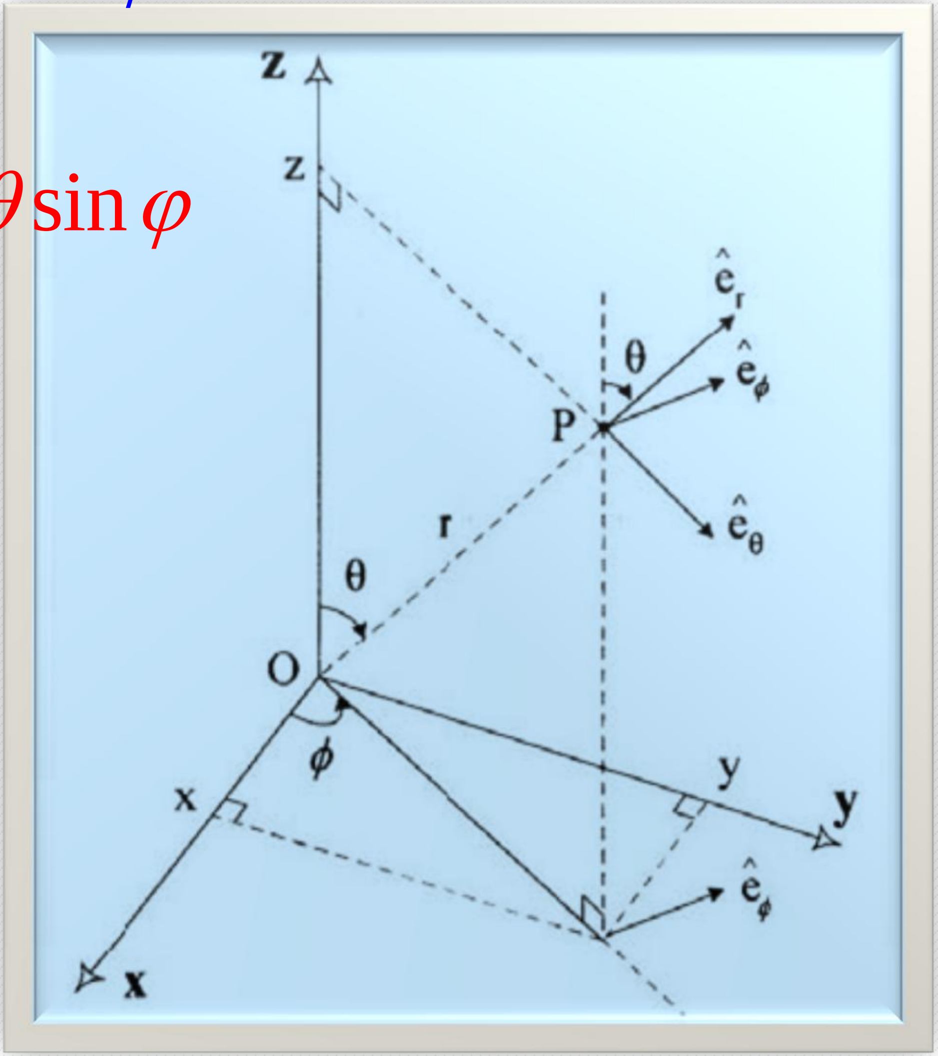
$$\hat{e}_r = \hat{i}\sin\theta\cos\varphi + \hat{j}\sin\theta\sin\varphi + \hat{k}\cos\theta$$

$$\hat{e}_\theta = \hat{i}\cos\theta\cos\varphi + \hat{j}\cos\theta\sin\varphi$$

$$-k\sin\theta$$

$$\hat{e}_\varphi = -\hat{i}\sin\varphi + \hat{j}\cos\varphi$$

$$\vec{r} = r\hat{e}_r$$



$$\vec{r} = r\hat{e}_r \qquad \vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{e}_r + r\frac{d\hat{e}_r}{dt}$$

$$\hat{e}_r = \hat{i}\sin\theta\cos\varphi + \hat{j}\sin\theta\sin\varphi + \hat{k}\cos\theta$$

$$\frac{d\hat{e}_r}{dt} = \frac{d\hat{e}_r}{d\theta}\frac{d\theta}{dt} + \frac{d\hat{e}_r}{d\varphi}\frac{d\varphi}{dt}$$

$$= \dot{\theta}(\hat{i}\cos\theta\cos\varphi + \hat{j}\cos\theta\sin\varphi - k\sin\theta)$$

$$+\dot{\varphi}\sin\theta(-\hat{i}\sin\varphi + \hat{j}\cos\varphi) = \dot{\theta}\hat{e}_\theta + \dot{\varphi}\sin\theta\hat{e}_\varphi$$

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\dot{\varphi}\sin\theta\hat{e}_\varphi$$

$$\frac{d\hat{e}_\theta}{dt} = -\dot{\theta}\hat{e}_r + \dot{\varphi}\cos\theta\hat{e}_\varphi \qquad \frac{d\hat{e}_\varphi}{dt} = -\dot{\varphi}\sin\theta\hat{e}_r - \dot{\varphi}\cos\theta\hat{e}_\theta$$

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\dot{\varphi}\sin\theta\hat{e}_\varphi \qquad \vec{a} = \frac{d\vec{v}}{dt}$$

$$\vec{a} = \ddot{r}\hat{e}_r + \dot{r}\frac{d\hat{e}_r}{dt} + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\frac{d\hat{e}_\theta}{dt}$$

$$+\dot{r}\dot{\varphi}\sin\theta\hat{e}_\varphi + r\ddot{\varphi}\sin\theta\hat{e}_\varphi + r\dot{\varphi}\sin\theta\frac{d\hat{e}_\varphi}{dt}$$

$$\vec{a} = \ddot{r}\hat{e}_r + \dot{r}(\dot{\theta}\hat{e}_\theta + \dot{\varphi}\sin\theta\hat{e}_\varphi)$$

$$+\dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}(-\dot{\theta}\hat{e}_r + \dot{\varphi}\cos\theta\hat{e}_\varphi)$$

$$+\dot{r}\dot{\varphi}\sin\theta\hat{e}_\varphi + r\ddot{\varphi}\sin\theta\hat{e}_\varphi + r\dot{\varphi}\sin\theta(-\dot{\varphi}\sin\theta\hat{e}_r - \dot{\varphi}\cos\theta\hat{e}_\theta)$$

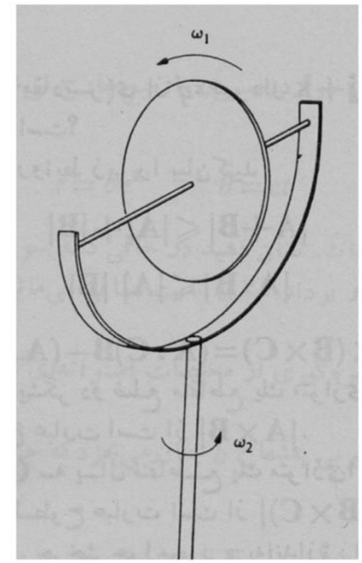
$$\vec{a} = (\ddot{r} - r\dot{\varphi}^2 \sin^2\theta - r\dot{\theta}^2)\hat{e}_r$$

$$+(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2 \sin\theta\cos\theta)\hat{e}_\theta$$

$$+(r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta)\hat{e}_\varphi$$

Example (2): $r = b, \quad \theta = \omega_1 t, \quad \varphi = \omega_2 t$

$$\dot{r} = \ddot{r} = 0, \quad \dot{\theta} = \omega_1, \quad \ddot{\theta} = 0, \quad \dot{\varphi} = \omega_2, \quad \ddot{\varphi} = 0$$



$$\vec{a} = (\ddot{r} - r\dot{\varphi}^2 \sin^2\theta - r\dot{\theta}^2)\hat{e}_r$$

$$+(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2 \sin\theta\cos\theta)\hat{e}_\theta$$

$$+(r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta)\hat{e}_\varphi$$

$$\vec{a} = (-b\omega_2{}^2 \sin^2\theta - b\omega_1{}^2)\hat{e}_r - b\omega_2{}^2 \sin\theta\cos\theta\,\hat{e}_\theta$$

$$+2b\omega_1\omega_2\cos\theta\,\hat{e}_\varphi$$

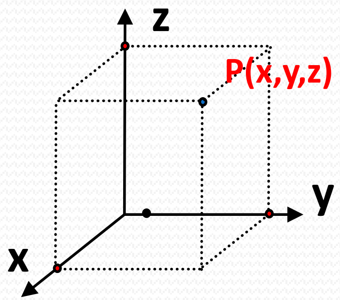
$$\vec{a} = -b\omega_1{}^2\hat{e}_r + 2b\omega_1\omega_2\hat{e}_\varphi$$

at the top Θ=0

Orthogonal Coordinate Systems: (coordinates mutually perpendicular)

Cartesian Coordinates

P (x,y,z)

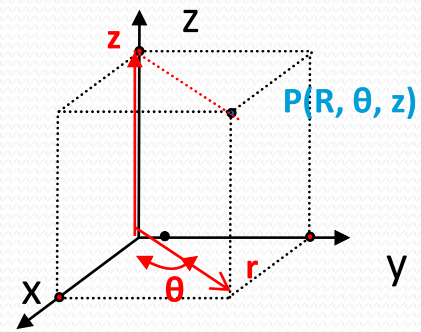


$$\vec{v} = \dot{x}\,\hat{i} + \dot{y}\,\hat{j} + \dot{z}\,\hat{k}$$

$$\vec{a} = \ddot{x}\,\vec{i} + \ddot{y}\,\vec{j} + \ddot{z}\,\vec{k}$$

Cylindrical Coordinates

P (r, Θ, z)

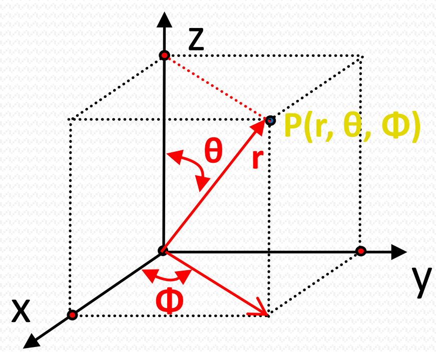


$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{R}\,\hat{e}_R + R\dot{\theta}\,\hat{e}_\theta + \dot{z}\hat{e}_z$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \left(\ddot{R} - R\dot{\theta}^2\right)\hat{e}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\hat{e}_\theta + \ddot{z}\,\hat{e}_z$$

Spherical Coordinates

P (r, Θ, Φ)



Problem 15 $$r = be^{kt} \qquad \theta = ct$$

Velocity & Acceleration in Plane Polar Coordinates

$$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta$$

$$\vec{v}, \quad \vec{a}, \qquad \left|\vec{v}\right|, \quad \left|\vec{a}\right|$$

$$\hat{v}.\hat{a} = va\cos\theta \rightarrow \cos\theta = \frac{\vec{v}.\vec{a}}{va}$$

Problem 22 $\left|\vec{v}\times\vec{a}\right| = v^3/\rho$

$$\vec{v} = v\,\hat{\tau}$$

$$\vec{a} = \dot{v}\,\hat{\tau} + \frac{v^2}{\rho}\hat{n}$$

$$\vec{v}\times\vec{a} = (v\,\hat{\tau})\times\left(\dot{v}\,\hat{\tau} + \frac{v^2}{\rho}\hat{n}\right) = (v\,\hat{\tau}\times v\,\hat{\tau}) + \left(v\,\hat{\tau}\times\frac{v^2}{\rho}\hat{n}\right)$$

$$= \frac{v^3}{\rho}(\hat{k}) \Rightarrow \left|\vec{v}\times\vec{a}\right| = v^3/\rho$$

Chapter 2

Newtonian Mechanics

Rectilinear Motion of a Particle

Rectilinear motion: position, velocity, and acceleration of a particle as it moves along a straight line.

Curvilinear motion: position, velocity, and acceleration of a particle as it moves along a curved line in two or three dimensions.

Newton's Laws of Motion

Newton's First Law. (Law of Inertia)

1. Newton's first law tells us:

Consider a body on which no net force acts.

1) If the body is at rest, it will remain at rest;

2) If the body is moving with constant velocity, it will continue to do so, no force is needed to keep it moving.

Inertia – The tendency of a body to maintain its state of rest or constant velocity.

Newton's first law is often called the law of inertia.

Inertial reference frames:

An inertial reference frame is one in which Newton’s first law is valid.

This excludes rotating and accelerating frames.

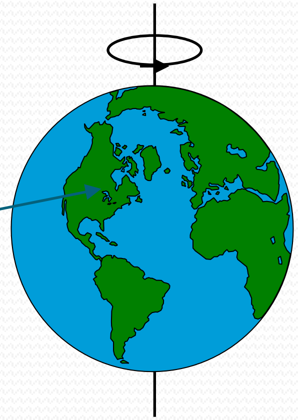
All accelerating reference frames are noninertial.

If no forces act, there is no acceleration

An IRF is a reference frame that is not accelerating (or rotating) with respect to the “fixed stars”.

Is Urbana a good IRF?

Is Urbana accelerating?



Urbana is on the Earth.
The Earth is rotating.

What is the centripetal acceleration of Urbana?

$$a_U = \frac{v^2}{R} = \left(\frac{2\pi}{T}\right)^2 R$$

T = 1 day = 8.64×10^4 sec,
$R \sim R_E = 6.4 \times 10^6$ meters

a_U = 0.034 m/s2 (~ 1/300 g)
Close enough to *0* that we will ignore it.
Urbana is a pretty good IRF.

Newton's Second Law

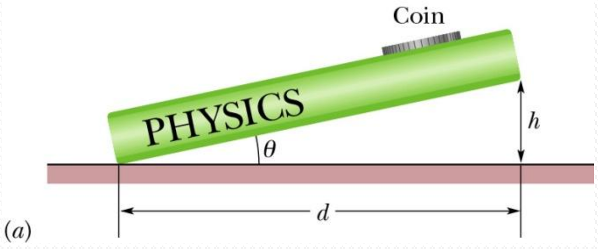
Newton's Second Law is about how force, mass, and acceleration are related.

The net force of an object is equal to the product of its mass and acceleration, or

$$F_{net} = \sum F_i = m\frac{d^2\vec{r}}{dt^2} = m\vec{a}$$

Mass is the measure of inertia of an object.

Mass is a property of an object.

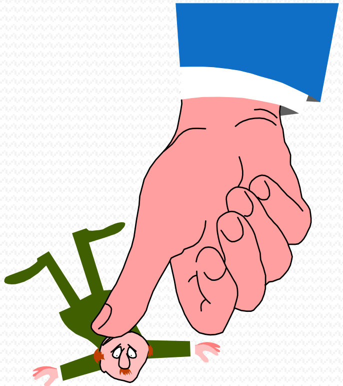


The Third Law addresses action – reaction pairs

3rd Law – For every action there is an equal and opposite reaction.

For every action there is an equal and opposite reaction.

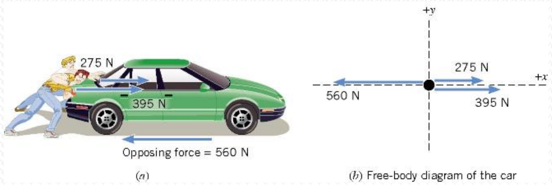
$$\vec{F}_{AB} = -\vec{F}_{BA}$$



Note: *the action and reaction forces always act on different bodies.*

Dynamical analysis using Newton's laws

In analyzing problems using Newton's law, there are several steps that we should follow:



1. choose a suitable *inertial reference frame*.

2. For each object in the problem, *draw a "free body diagram"*, showing all of the forces acting on that body.

3. For each body, *find the vector sum of all the forces*. In practice, this usually means separately adding the *x*, *y*, *z* components of the forces. Then use Eqs to find acceleration components

Linear Momentum

The linear momentum p of an object of mass m with a velocity of v is

$$\vec{p} = m\vec{v}$$

The Second Law can be writen as $\dfrac{d\vec{p}}{dt} = \vec{F}$

Force equals to the variation rate of momentum

Alternate Statement of the Second Law

$$\frac{d(m_A \overrightarrow{v_A})}{dt} = -\frac{d(m_B \overrightarrow{v_B})}{dt}$$

$$\frac{d}{dt}(\vec{p}_A + \vec{p}_B) = 0 \qquad (\vec{p}_A + \vec{p}_B) = \text{constant}$$

The momentum of a system keep invariant when there is no external force on it or the total external force is zero

Rectilinear Motion.

Uniform Acceleration Under a Constant Force

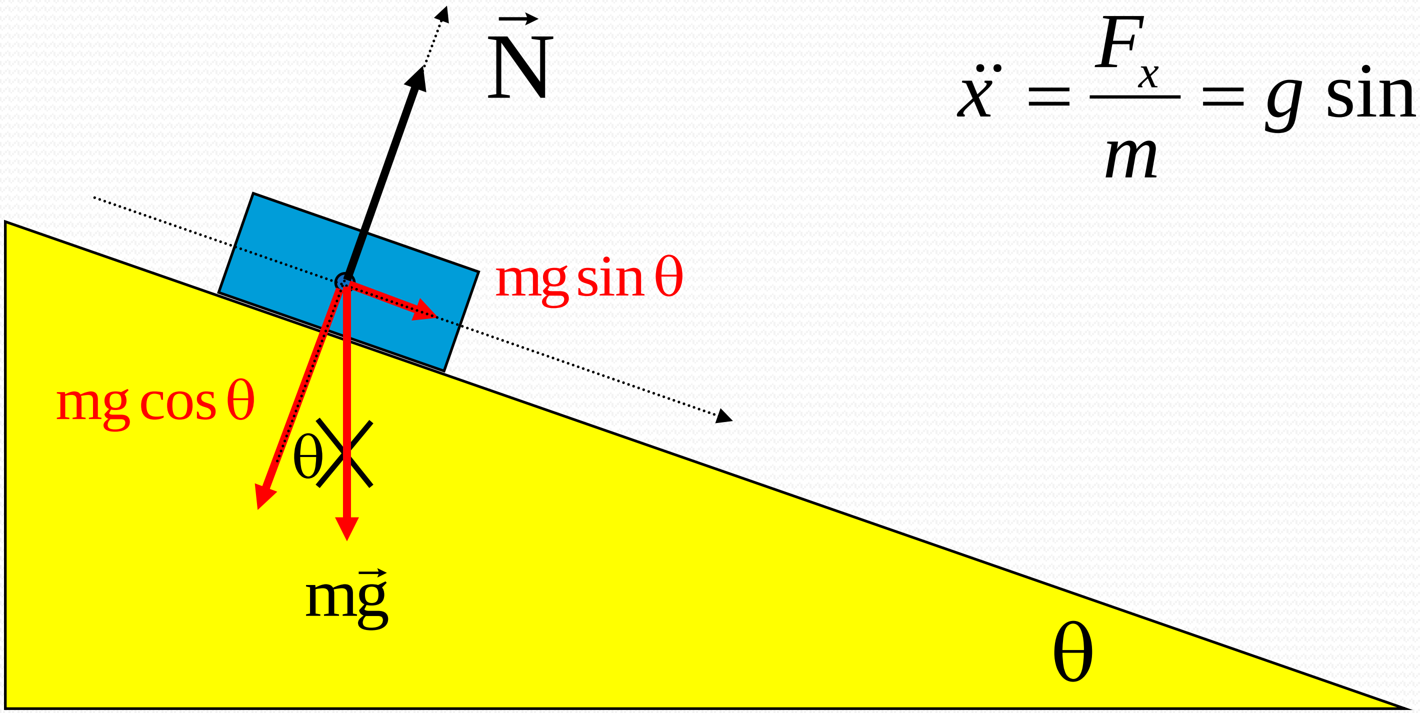
$$F_x(x, \dot{x}, t) = m\ddot{x}$$

$$\ddot{x} = \frac{dv}{dt} = \frac{F}{m} = \text{const.} = a$$

$$\dot{x} = v = at + v_0 \qquad x = \frac{1}{2}at^2 + v_0 t + x_0$$

$$2a(x - x_0) = v^2 - v_0^2$$

$$\ddot{x} = \frac{F_x}{m} = g\sin\theta$$



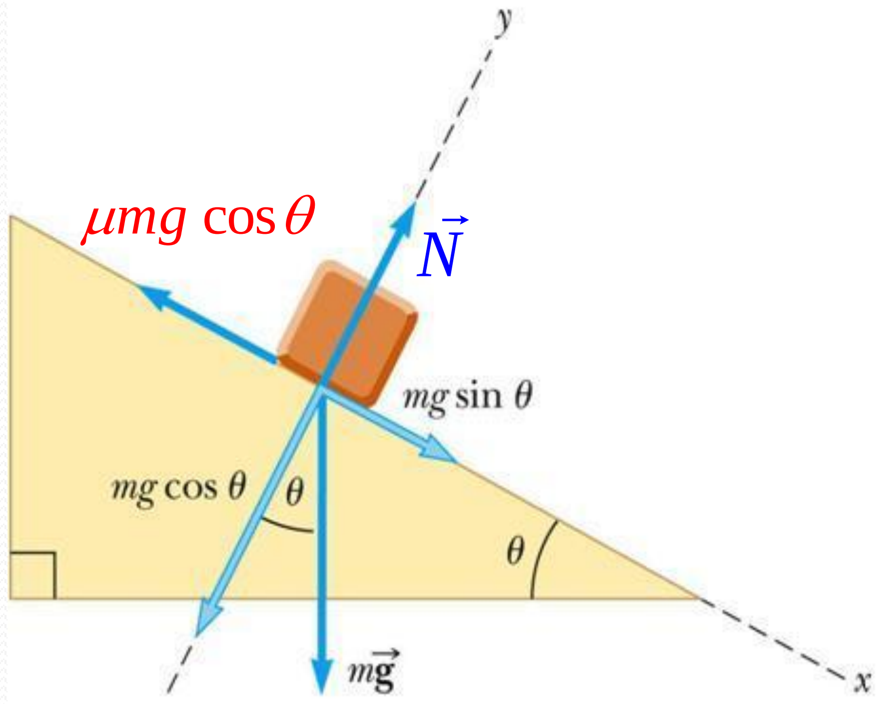
Block on a smooth incline plane

$$F_{net,x} = mg\sin\theta - f$$

$$f = \mu N = \mu mg\cos\theta$$

$$mg\sin\theta - \mu mg\cos\theta$$

$$\ddot{x} = \frac{F_x}{m} = g(\sin\theta - \mu\cos\theta)$$



$$\theta > \tan^{-1}\mu = \varepsilon$$

$$\theta = \varepsilon \rightarrow a = 0$$

$\theta < \varepsilon \rightarrow a$ is Neg.