مکانیک تحلیلی 1

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Position Vector of a Particle

Velocity and Acceleration in Rectangular Coordinates

A system for assigning coordinates, to the location of point in a reference frame

Cartesian coordinates = rectangular coordinates

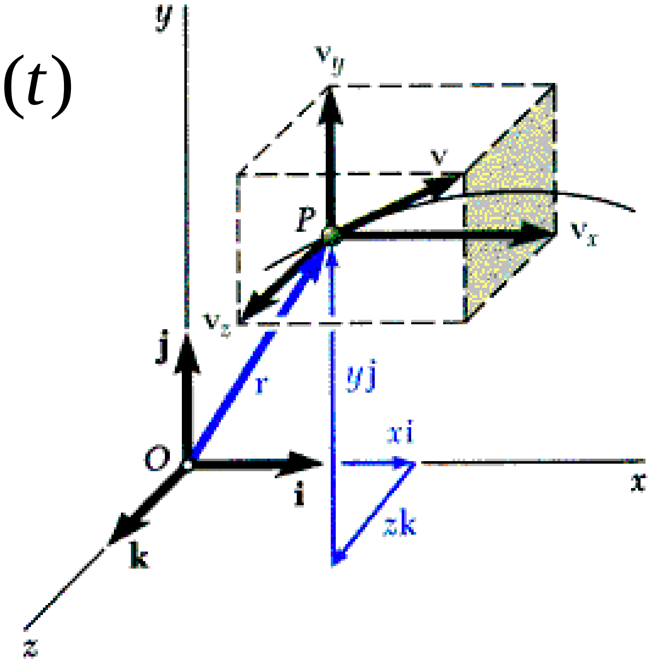
$$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z \qquad \Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t)$$

$$x = x(t) \qquad y = y(t) \qquad z = z(t)$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$= \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

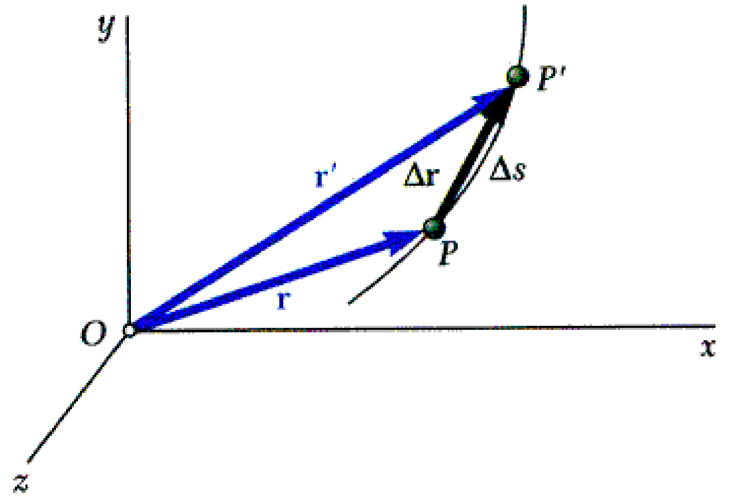
$$v = |\vec{v}| = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}$$

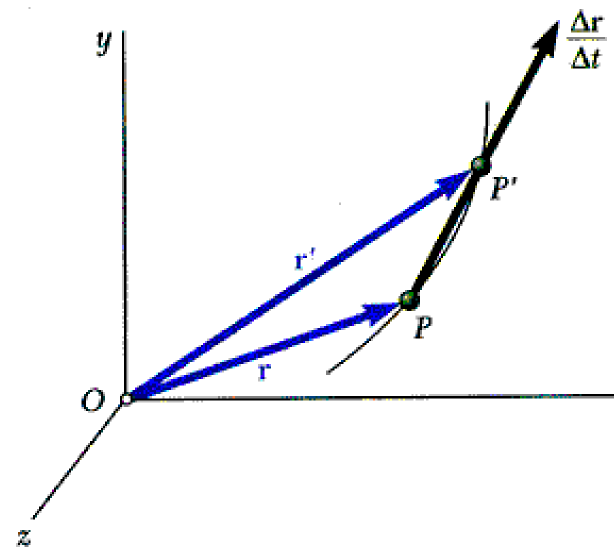


Particle moving along a curve other than a straight line is in *curvilinear motion.*

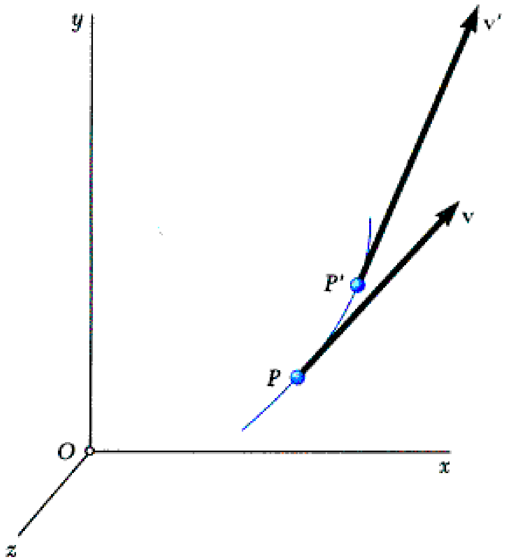
$$v = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

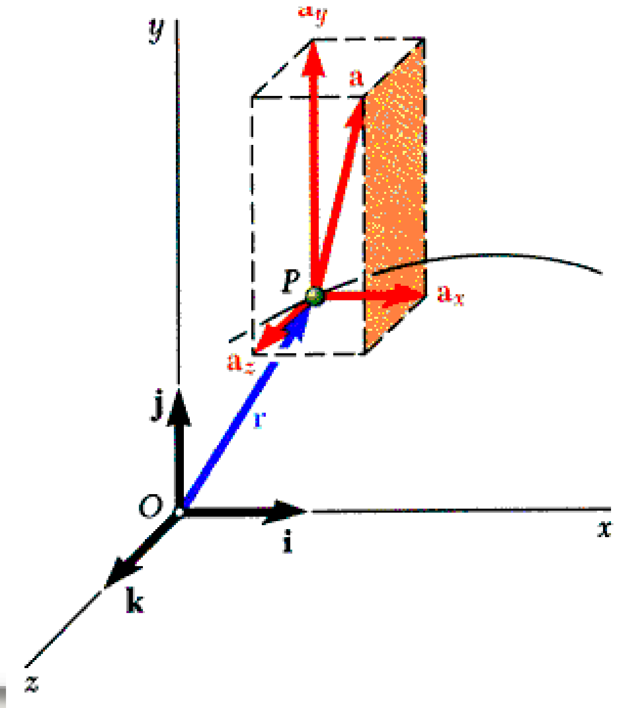
$$= \lim_{\Delta t \to 0} \frac{\left[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right]^{1/2}}{\Delta t}$$





Acceleration vector





$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

$$= \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}$$

$$= \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}$$

$$= a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$

Projectile motion

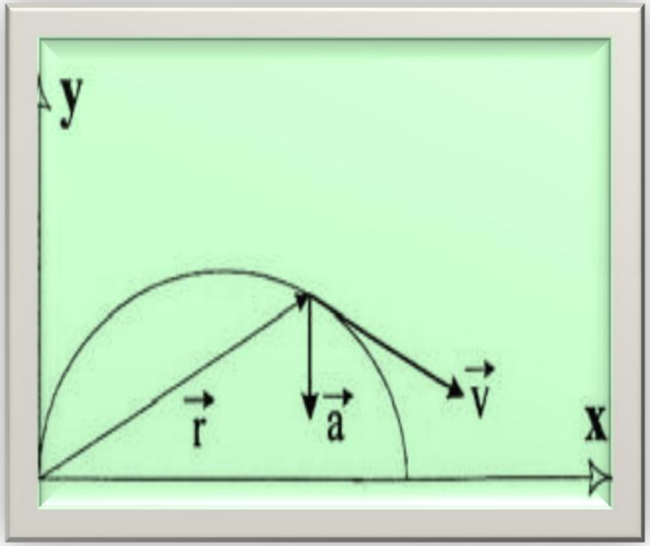
$$\vec{r}(t) = \hat{i}bt + j(ct - \frac{gt^2}{2}) + \hat{k}(0)$$

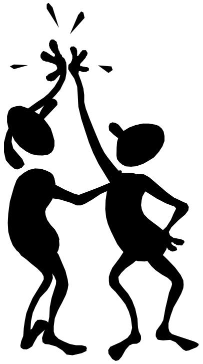
$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i}b + \hat{j}(c - gt)$$

$$\vec{a} = \frac{d\vec{v}}{dt} = -\hat{j}g$$

$$v = [b^2 + (c - gt)^2]^{1/2}$$

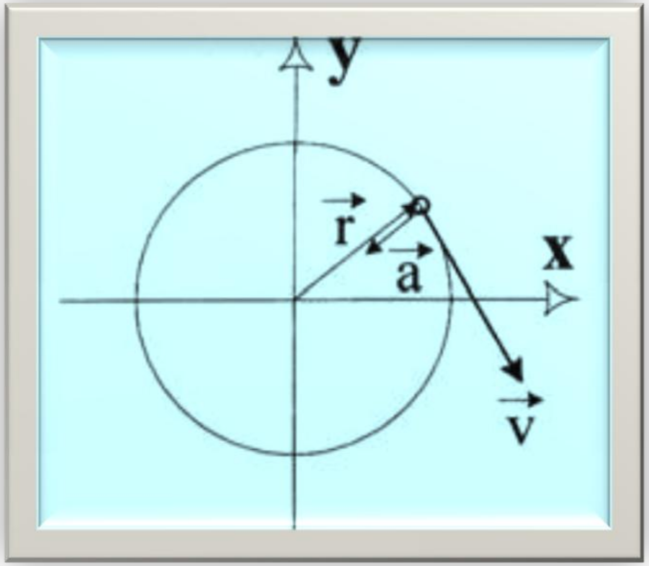
The path of motion is a parabola





Circular Motion

$$\vec{r} = \hat{i}\, b \sin \omega t + \hat{j}\, b \cos \omega t$$



$$\left|\vec{r}\right| = r$$

$$= (b^2 \sin^2 \omega t + b^2 \cos^2 \omega t)^{1/2} = b$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i}\, b\omega \cos \omega t - \hat{j}\, b\omega \sin \omega t$$

$$\vec{a} = -\omega^2 \vec{r}$$

$$v = \left|\vec{v}\right| = (b^2\omega^2 \cos^2 \omega t + b^2\omega^2 \sin^2 \omega t)^{1/2} = b\omega$$

$$\vec{a} = \frac{d\vec{v}}{dt} = -\hat{i}\, b\omega^2 \sin \omega t - \hat{j}\, b\omega^2 \cos \omega t$$

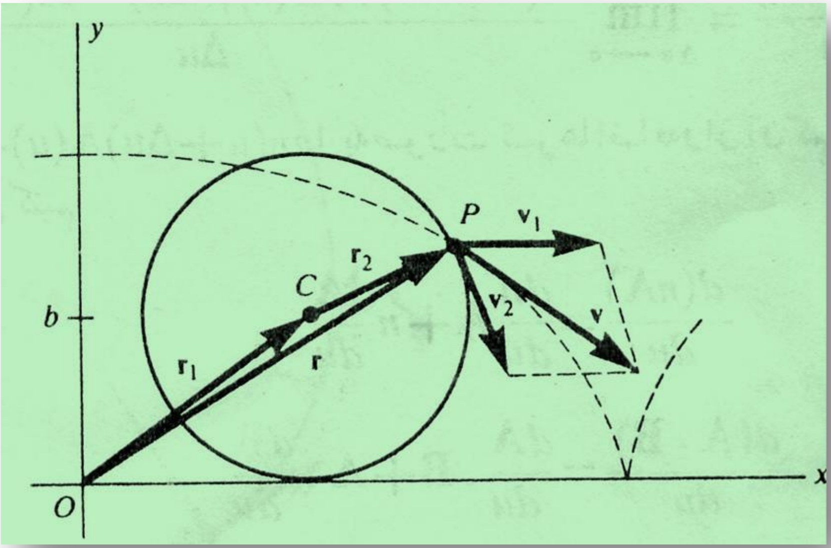
$$a = \left|\vec{a}\right| = b\omega^2$$

$$\vec{v}.\vec{a} = (b\omega\cos\omega t)(-b\omega^2 \sin\omega t)$$
$$+(-b\omega\sin\omega t)(-b\omega^2 \cos\omega t) = 0$$

$$\vec{r}.\vec{v} = 0$$

Rolling Wheel

$$\vec{r} = \vec{r}_1 + \vec{r}_2$$



$$\vec{r}_1 = \hat{i}\, b\omega t + \hat{j} b$$

$$v = r\omega$$

$$x = vt = r\omega t$$

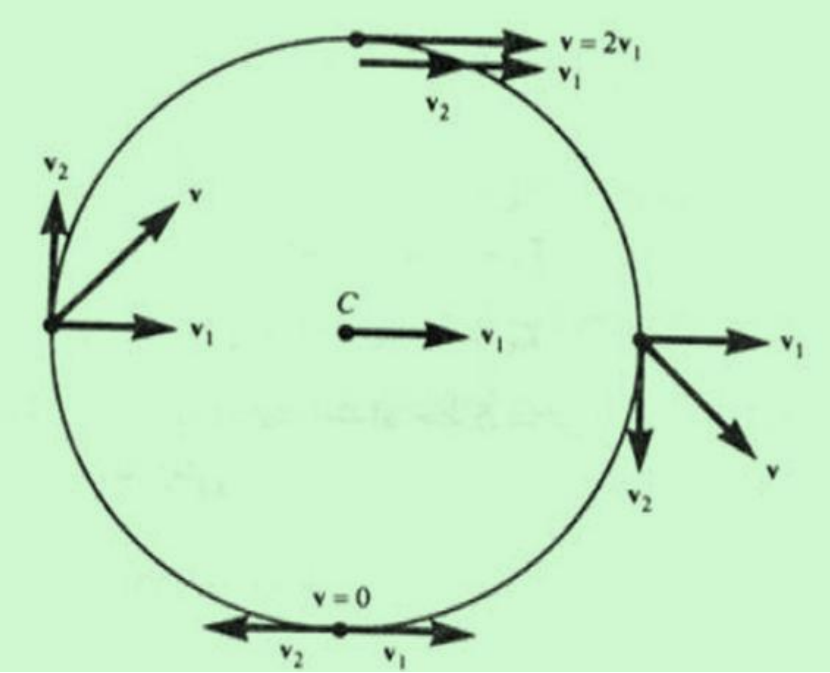
$$\vec{r}_2 = \hat{i}\, b \sin \omega t + \hat{j} b \cos \omega t$$

$$\vec{v}_1 = \frac{d\vec{r}_1}{dt} = \hat{i}\, b\omega$$

$$\vec{v}_2 = \hat{i}\, b\omega \cos \omega t - \hat{j} b\omega \sin \omega t$$

$$\vec{v} = \vec{v}_1 + \vec{v}_2 =$$

$$\hat{i}(b\omega + b\omega\cos\omega t) - \hat{j}b\omega\sin\omega t$$



$$for\ \ \omega t = 0, 2\pi, 4\pi, \ldots$$

$$\vec{v} = 2b\omega\hat{i}$$

$$for\ \ \omega t = \pi, 3\pi, 5\pi, \ldots$$

$$\vec{v} = 0$$

$$for\ \ \omega t = 2k\pi + \pi/2$$

$$\vec{v} = b\omega\hat{i} - b\omega\hat{j}$$

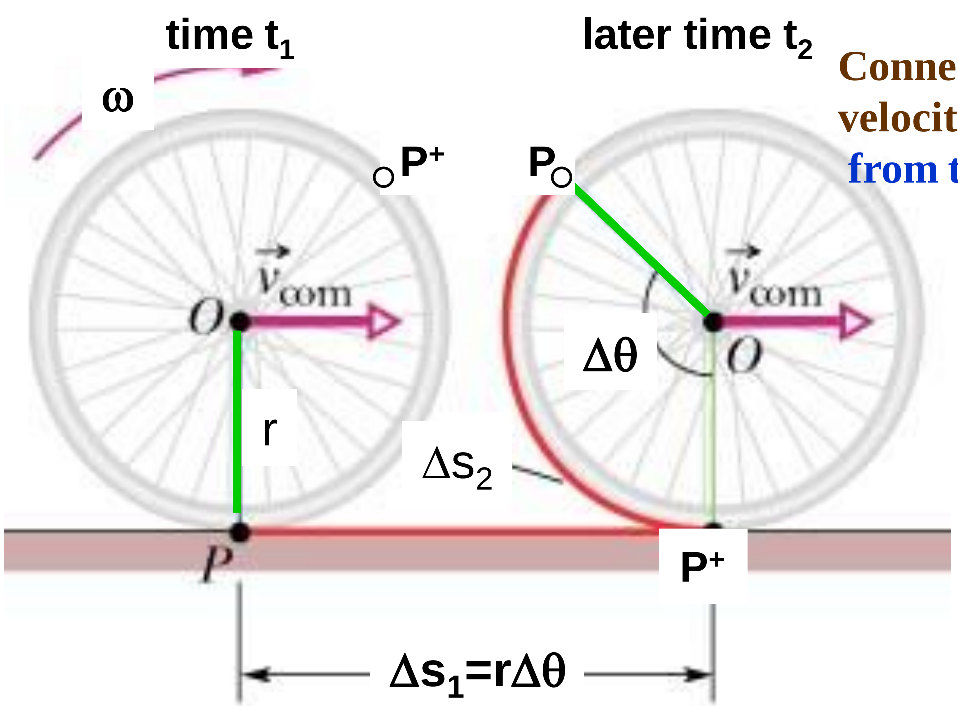
$$for\ \ \omega t = 2k\pi + 3\pi/2$$

$$\vec{v} = b\omega\hat{i} + b\omega\hat{j}$$

Rolling without slipping

What does it mean for a wheel to roll rather than slide?

- "No slipping" means
  - contact point "P" is stationary, o/w it would be sliding
  - friction must be *static* friction; as usual $f_s \leq \mu_s N$
  - distance covered Δs_1 must = arc swept out Δs_2 as wheel rotates by $\Delta\theta$
- Mass center moves along while wheel rotates around an axle (axis, cm)
- Friction at the point of contact tries to "match" the rotation rate to the mass center speed, providing torque.



Connection between constant mass center velocity and the angular velocity of the wheel

from the translational motion:

$$\Delta s_1 = v_{cm}\,\Delta t = \Delta s_2$$

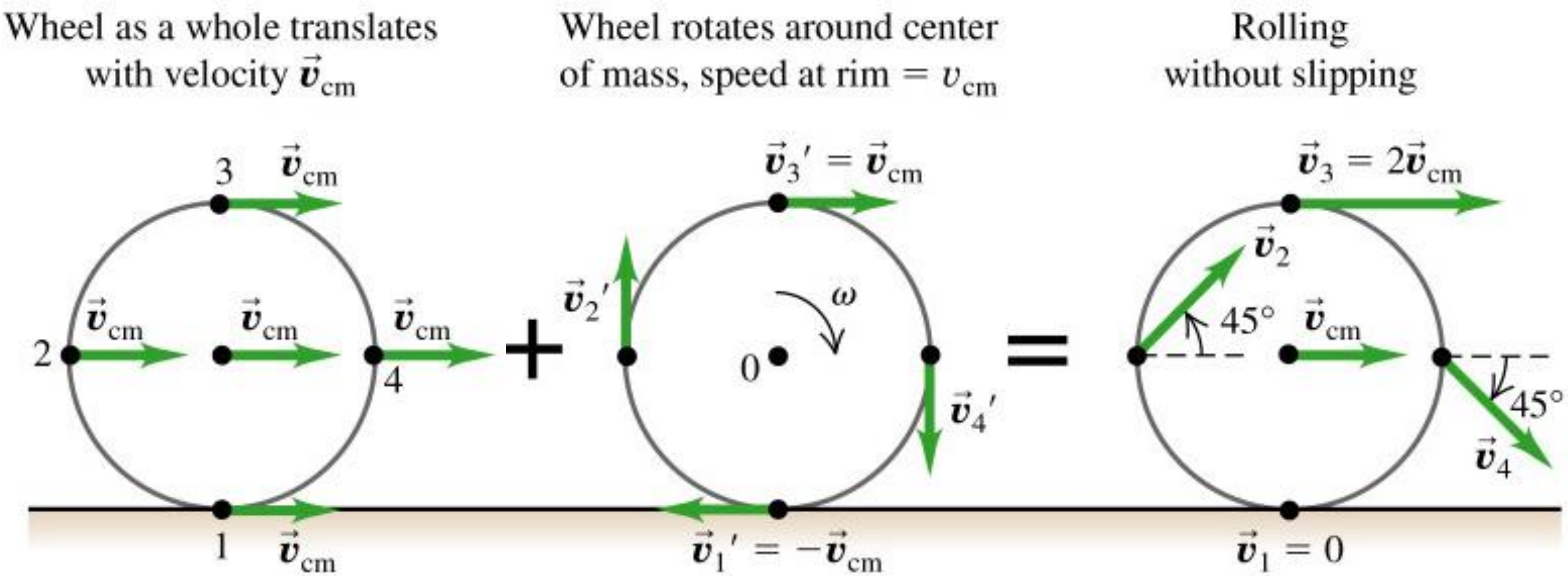
from the rotation:

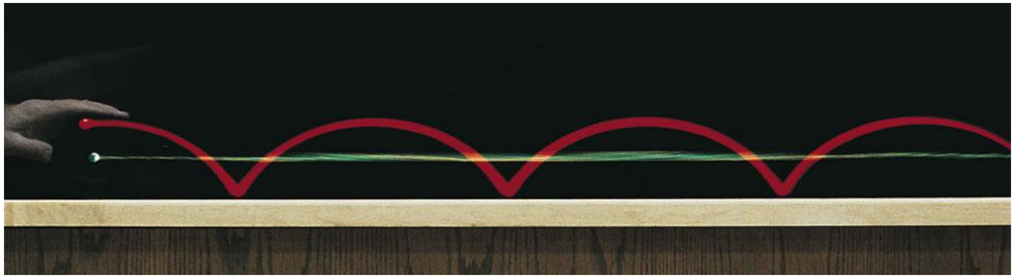
$$\Delta s_2 = r\,\Delta\theta = r\,\omega\,\Delta t$$

NO SLIPPING IMPLIES:

$$v_{cm} = r\omega = v_{\tan g}$$

A Wheel Rolling without Slipping



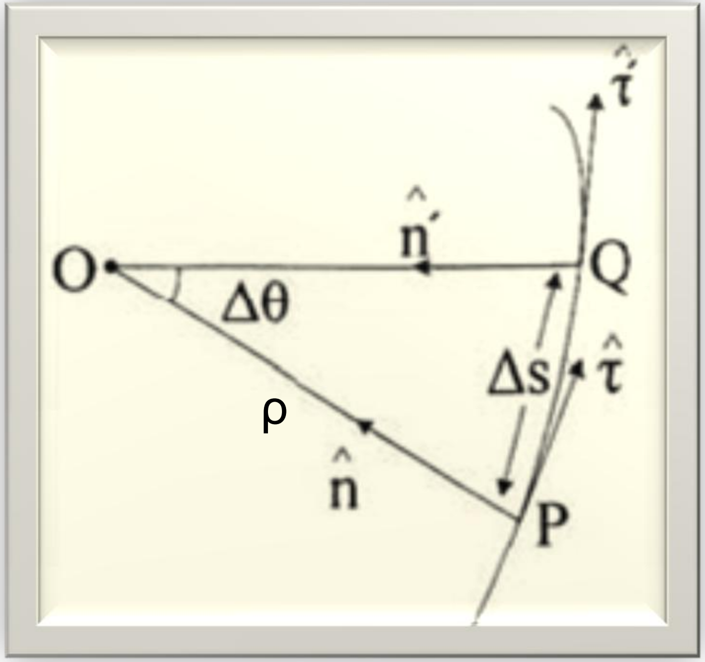


A cycloid is the locus of a point on the circumference of a circle rotating along a fixed line . . . .

Tangential and Normal Components of Acceleration

$$\vec{v} = v\hat{\tau}$$

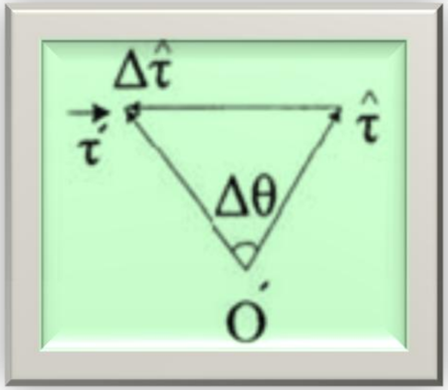
The velocity vector of a moving particle



$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}\hat{\tau} + v\frac{d\hat{\tau}}{dt}$$

$$\frac{d\hat{\tau}}{d\theta} = \hat{n}$$

$$\frac{d\hat{\tau}}{dt} = \frac{d\hat{\tau}}{d\theta}\frac{d\theta}{dt} = \hat{n}\frac{d\theta}{ds}\frac{ds}{dt} = \hat{n}\frac{v}{\rho}$$



$$\rho\, d\theta = ds \qquad \frac{ds}{dt} = v$$

$$\vec{a} = \dot{v}\hat{\tau} + \frac{v^2}{\rho}\hat{n}$$

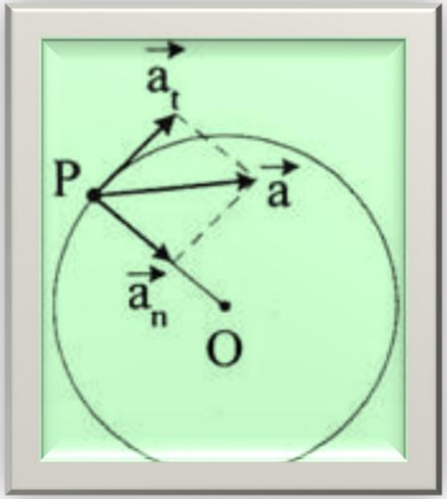
$$a_\tau = \dot{v} = \ddot{s}$$

tangential acceleration

$$a_n = \frac{v^2}{\rho}$$

normal component
Centripetal acceleration

$$|\vec{a}| = \left|\frac{d\vec{v}}{dt}\right| = (\dot{v}^2 + \frac{v^4}{\rho^2})^{1/2}$$

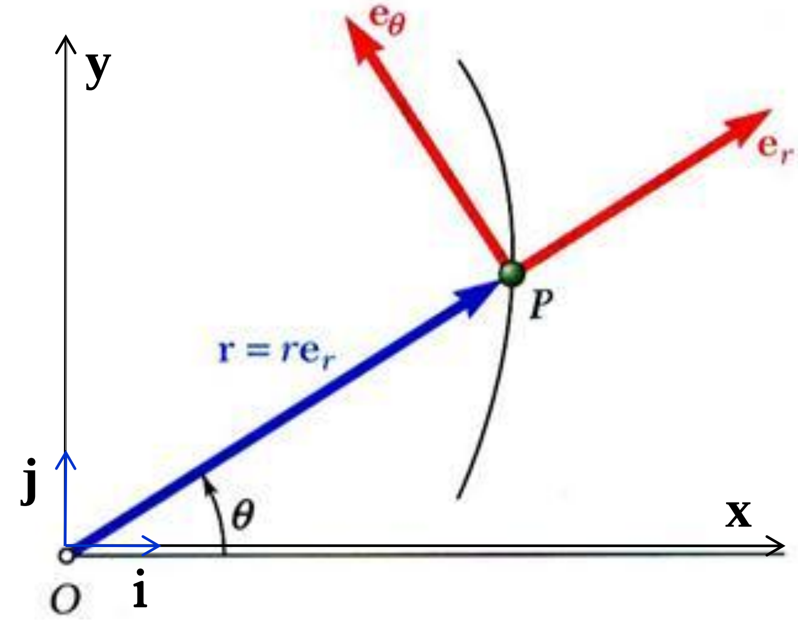


Velocity & Acceleration in Plane Polar Coordinates

$$\vec{r} = r\hat{e}_r$$

$$\vec{v} = \frac{d}{dt}\left(r\hat{e}_r\right) = \dot{r}\hat{e}_r + r\frac{d\hat{e}_r}{dt}$$

$$= \dot{r}\hat{e}_r + r\frac{d\hat{e}_r}{d\theta}\frac{d\theta}{dt}$$

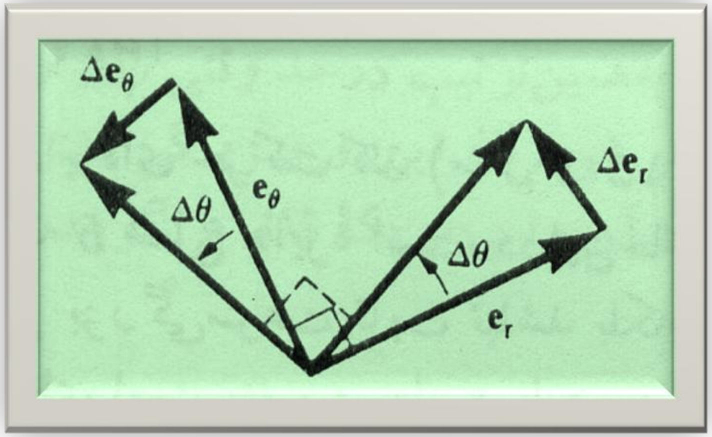


$$\hat{e}_r = \hat{i}\cos\theta + \hat{j}\sin\theta$$

$$\hat{e}_\theta = \hat{i}\cos(\pi/2+\theta) + \hat{j}\sin(\pi/2+\theta) = -\hat{i}\sin\theta + \hat{j}\cos\theta$$

$$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

$$\left|\Delta\hat{e}_r / 2\right| = \sin(\Delta\theta / 2)$$



$$\lim_{\Delta\theta\to 0}\frac{\Delta\hat{e}_r}{\Delta\theta} = \lim_{\Delta\theta\to 0}\frac{\sin(\Delta\theta/2)}{\Delta\theta/2}\hat{e}_\theta$$

$$\Delta\hat{e}_\theta \cong -\hat{e}_r\Delta\theta$$

$$\frac{d\vec{e}_r}{d\theta} = \hat{e}_\theta$$

$$\frac{d\vec{e}_\theta}{dt} = -\hat{e}_r\frac{d\theta}{dt}$$

$$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

Similarly, the particle acceleration vector is

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta\right)$$

$$\vec{a} = \ddot{r}\hat{e}_r + \dot{r}\frac{d\hat{e}_r}{dt} + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\frac{d\hat{e}_\theta}{dt}$$

$$\frac{d\vec{e}_r}{dt} = \hat{e}_\theta\frac{d\theta}{dt} \qquad \frac{d\vec{e}_\theta}{dt} = -\hat{e}_r\frac{d\theta}{dt}$$

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta$$

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta$$

$$a_r = \ddot{r} - r\dot{\theta}^2$$ radial component

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$ transverse component

SPECIAL CASES OF MOTION

1) If a particle moves on a circle of constant radius

$$\vec{a} = \left(-r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta}\right)\vec{e}_\theta$$

$$\begin{cases} a_r = r\dot{\theta}^2 \\ a_\theta = r\ddot{\theta} \end{cases}$$

2) If Θ is constant

$$\vec{a} = \ddot{r}\,\vec{e}_r$$

3) If r & ω are constant

$$\vec{a} = -r\dot{\theta}^2\,\vec{e}_r$$

Example: A honeybee in on its hive in a spiral path in such a way that the radial distance decreases a constant rate: $r = b - ct$
While the angular speed increases at a constant rate: $\dot{\theta} = kt$
Find the speed as a function of time.

$$\dot{r} = -c \quad , \quad \ddot{r} = 0$$

$$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

$$\vec{v} = -c\,\hat{e}_r + (b - ct)kt\,\hat{e}_\theta$$

$$v = \left[c^2 + (b - ct)^2 k^2 t^2\right]^{1/2}$$

$$t = 0, \quad r = b$$

$$t = b/c, \quad r = 0, \quad v = c$$

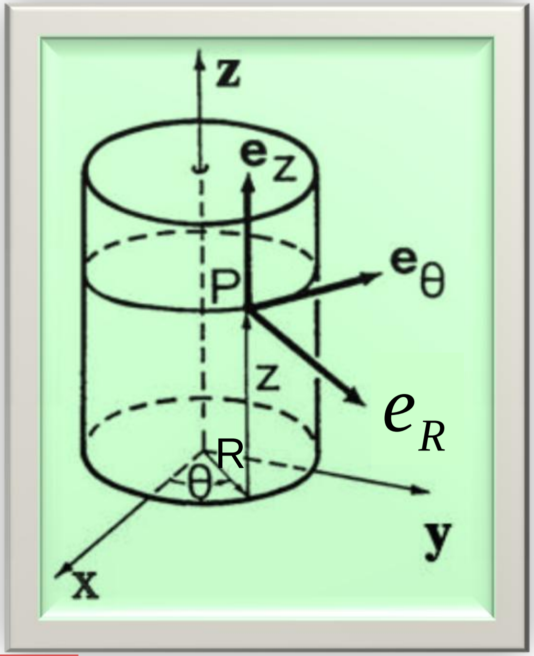
$$t \leq b/c$$

Velocity & Acceleration in Cylindrical Coordinate

R radial distance in x-y plane $0 \leq R \leq \infty$

θ azimuth angle measured from the positive x-axis $0 \leq \theta < 2\pi$

z $-\infty < z < \infty$



$$\hat{e}_R \times \hat{e}_\theta = \hat{e}_z$$

$$\hat{e}_\theta \times \hat{e}_z = \hat{e}_R$$

$$\hat{e}_z \times \hat{e}_R = \hat{e}_\theta$$

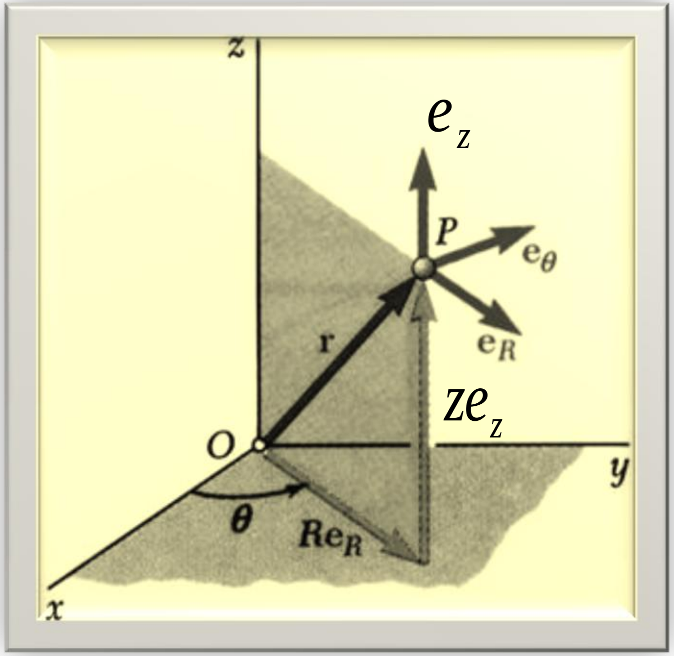
$$\hat{e}_R = \hat{i}\cos\theta + \hat{j}\sin\theta$$

$$\hat{e}_\theta = -\hat{i}\sin\theta + \hat{j}\cos\theta$$

$$\hat{e}_z = \hat{k}$$

$$x = R\cos\theta$$
$$y = R\sin\theta$$
$$z = z$$

$$R = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}(y/x)$$
$$z = z$$



Position vector, $\vec{r} = R\,\hat{e}_R + z\,\hat{e}_z$

$$d\hat{e}_R / dt = \hat{e}_\theta \dot{\theta} \qquad d\hat{e}_\theta / dt = -\hat{e}_R \dot{\theta}$$

Velocity vector, $\vec{v} = \dfrac{d\vec{r}}{dt} = \dot{R}\,\hat{e}_R + R\dot{\theta}\,\hat{e}_\theta + \dot{z}\hat{e}_z$

Acceleration vector, $\vec{a} = \dfrac{d\vec{v}}{dt} = \left(\ddot{R} - R\dot{\theta}^2\right)\hat{e}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\hat{e}_\theta + \ddot{z}\,\hat{e}_z$