ANALYTICAL MECHANICS 1

Lecture 19

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Orbital Parameters from the Conditions at Closest Approach

$$e = \frac{mh^2}{kr_\circ} - 1 \qquad h = r^2\dot{\theta} \xrightarrow{r=r_\circ} h = r_\circ^{\,2}\dot{\theta}_\circ = r_\circ v_\circ$$

$$e = \frac{mr_\circ v_\circ^{\,2}}{k} - 1 \qquad if \;\; e = 0 \rightarrow k = mr_\circ v_\circ^{\,2}$$

$$\frac{k}{r_\circ^{\,2}} = \frac{mv_\circ^{\,2}}{r_\circ} \rightarrow \frac{k}{mr_\circ} = v_c^{\,2} \quad if \;\; v_\circ = v_c \rightarrow cir.\ \mathrm{orb.}$$

$$e = \frac{mr_\circ v_\circ^{\,2}}{k} - 1$$

$$for \; v_o \geq v_c. \qquad e = (v_\circ / v_c)^2 - 1$$

$$r = r_{\circ}\frac{1+e}{1+e\cos\theta}$$ *Equation of the orbit*

$$r = r_{\circ}\frac{(v_{\circ}/v_{c})^{2}}{1+[(v_{\circ}/v_{c})^{2}-1]\cos\theta}$$

$if\ \theta = \pi \rightarrow$ $$r_{1} = r_{\circ}\frac{(v_{\circ}/v_{c})^{2}}{2-(v_{\circ}/v_{c})^{2}}$$

$if\ \theta = 0 \rightarrow$ $$r_{1} = r_{\circ}$$

$Example\ 6:$

$a)\ k = GM_e m$ $\qquad v_c{}^2 = \frac{k}{mr_\circ} = \frac{GM_e}{r_\circ}$

$mg = GM_e m / R_e{}^2$ $\qquad GM_e = gR_e{}^2$

$v_c = \left(\frac{gR_e{}^2}{r_\circ}\right)^{1/2}$ $\qquad r_\circ \simeq R_e$

$v_c = (gR_e)^{1/2} = (9.8ms^{-2} \times 6.4 \times 10^6 m)^{1/2} \simeq 8\,km/s$

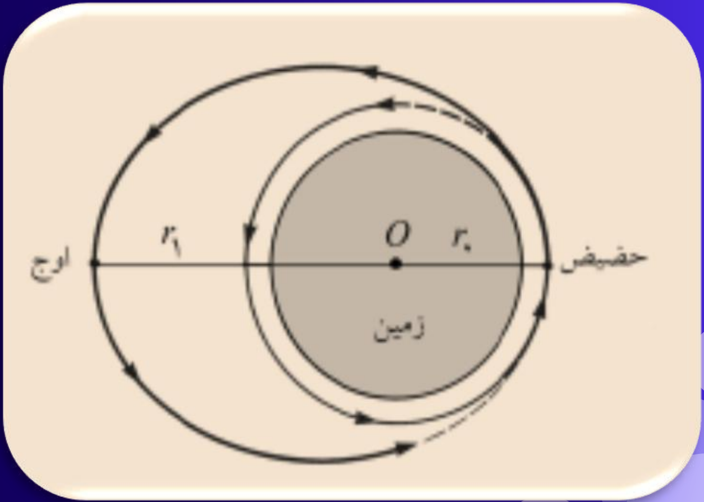
$b)\ v_\circ = 1.15 v_c$ $\qquad r = r_\circ \frac{(v_\circ / v_c)^2}{1 + [(v_\circ / v_c)^2 - 1]\cos\theta}$

$$r = r_\circ \frac{(1.15)^2}{1+[(1.15)^2-1]\cos\theta} = r_\circ \frac{1.3225}{1+0.3225\cos\theta}$$

$$r_1 = r_\circ \frac{(v_\circ/v_c)^2}{2-(v_\circ/v_c)^2} \qquad if \quad v_c = v_\circ \rightarrow r_1 = r_\circ$$

$$r_1 = r_\circ \frac{(1.15)^2}{2-(1.15)^2}$$

$$= r_\circ \frac{1.3225}{2-1.3225} = 1.95 r_\circ$$



Orbital Energies in the Inverse-square Field

$$f(r) = -\frac{k}{r^2} \qquad V(r) = -\frac{k}{r} = -ku$$

$$\frac{1}{2}mh^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] + V(u^{-1}) = E$$

$$\frac{1}{2}mh^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] - ku = E$$

$$\left(\frac{du}{d\theta}\right)^2 = \frac{E + ku}{\frac{1}{2}mh^2} - u^2$$

$$\frac{du}{d\theta} = \left(\frac{2E + 2ku}{mh^2} - u^2\right)^{1/2}$$

$$\int \frac{du}{\left(\frac{2E}{mh^2} + \frac{2ku}{mh^2} - u^2\right)^{1/2}} = \int d\theta$$

$$\int \frac{du}{(a^2 - u^2)^{1/2}} = \sin^{-1}\frac{u}{a}$$

$$\theta = \sin^{-1}\left[\frac{mh^2u - k}{(k^2 + 2Emh^2)^{1/2}}\right] + \theta_\circ$$

$$\theta = \sin^{-1}\left[\frac{mh^2u - k}{(k^2 + 2Emh^2)^{1/2}}\right] + \theta_\circ$$

$$\theta_\circ = -\pi/2 \qquad \sin(\theta - \theta_0) = \sin(\theta + \pi/2) = \cos\theta$$

$$\cos\theta = \frac{mh^2u - k}{(k^2 + 2Emh^2)^{1/2}}$$

$$u = \frac{1 + (1 + 2Emh^2k^{-2})^{1/2}\cos\theta}{mh^2/k}$$

$$r = \frac{mh^2k^{-1}}{1 + (1 + 2Emh^2k^{-2})^{1/2}\cos\theta} \xrightarrow{compare} r = r_\circ \frac{1+e}{1+e\cos\theta}$$

$$e = (1 + 2Emh^2k^{-2})^{1/2}$$

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$E < 0 \quad e < 1$ *closed orbits (ellipse or circle)*

$E = 0 \quad e = 1$ *parabolic orbit*

$E > 0 \quad e > 1$ *hyperbolic orbit*

$$E = T + V$$

$$T < |V| \rightarrow E < 0 \rightarrow \text{closed orbits}$$

$$T \geq |V| \rightarrow E \geq 0 \rightarrow \text{open orbits}$$

$$k = GMm$$

$$E = T + V$$

$$\frac{mv^2}{2} - \frac{GMm}{r} = E = const\cdot$$

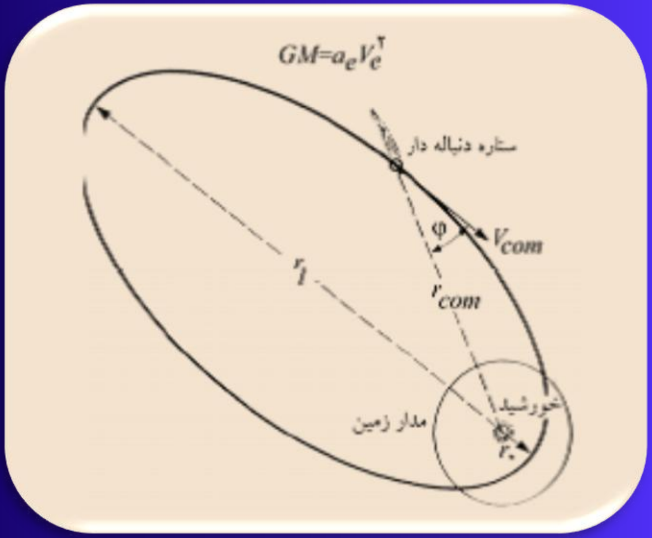
$$\text{if} \quad v^2 < \frac{2GM}{r} \rightarrow E < 0 \rightarrow ellipse$$

$$\text{if} \quad v^2 = \frac{2GM}{r} \rightarrow E = 0 \rightarrow parabola$$

$$\text{if} \quad v^2 > \frac{2GM}{r} \rightarrow E > 0 \rightarrow hyperbola$$

example 7:

$$e = (1 + 2Emh^2k^{-2})^{1/2}$$



$$k = GMm \qquad \frac{1}{2}mv^2 - \frac{GMm}{r} = E$$

$$h^2 = \left|\vec{r} \times \vec{v}\right|^2 = (r_{com} v_{com} \sin\varphi)^2$$

$$e = \left[1 + \left(v^2_{com} - \frac{2GM}{r_{com}}\right)\left(\frac{r_{com} v_{com} \sin\varphi}{GM}\right)^2\right]^{1/2}$$

$$GM = a_e v_e^{\,2}$$

$$e = \left[1 + \left(V^2 - \frac{2}{R}\right)(RV \sin\varphi)^2\right]^{1/2}$$

$$V = \frac{v_{com}}{v_e} \qquad R = \frac{r_{com}}{a_e}$$

$$if \quad v_{com} = \frac{1}{2} v_e \quad r_{com} = 4r \qquad \varphi = 30^0$$

$$V = 0.5 \qquad R = 4$$

$$e = \left[1 + (V^2 - \frac{2}{R})(RV \sin\varphi)^2 \right]^{1/2}$$

$$e = \left[1 + (0.25 - 0.5)(4 \times 0.5 \times 0.5)^2 \right]^{1/2} = (0.75)^{1/2} = 0.866$$