مکانیک تحلیلی 1

درس دوم

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Triplet product

$$\vec{A}.(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Properties

Interchanging any two rows reverses the sign of the determinant, so

$$\vec{\mathrm{A}}.(\vec{\mathrm{B}}\times\vec{\mathrm{C}}) = -\vec{\mathrm{A}}.(\vec{\mathrm{C}}\times\vec{\mathrm{B}})$$

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Interchanging rows twice the original sign is restored, so

$$\vec{\mathrm{A}}.(\vec{\mathrm{B}}\times\vec{\mathrm{C}}) = \vec{\mathrm{B}}.(\vec{\mathrm{C}}\times\vec{\mathrm{A}}) = \vec{\mathrm{C}}.(\vec{\mathrm{A}}\times\vec{\mathrm{B}})$$

If any two vectors of the scalar triple product are equal, the scalar triple product is zero.

$$\vec{A}.(\vec{A}\times\vec{C}) = 0$$

Vector Triple Product

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}.\vec{C})-\vec{C}(\vec{A}.\vec{B})$$

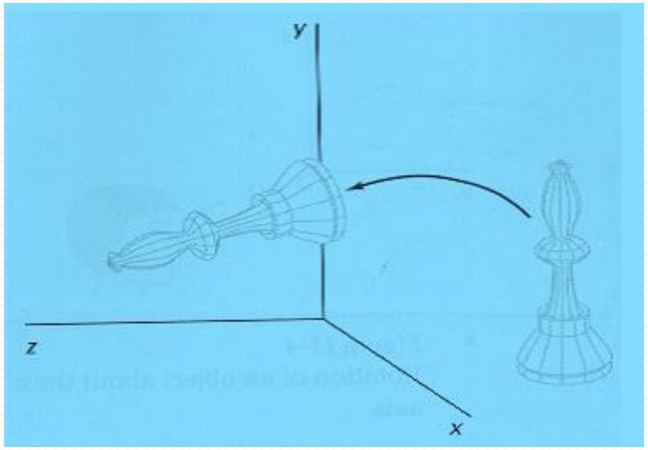
$$(\vec{A}\times\vec{B})\times\vec{C}$$

Example

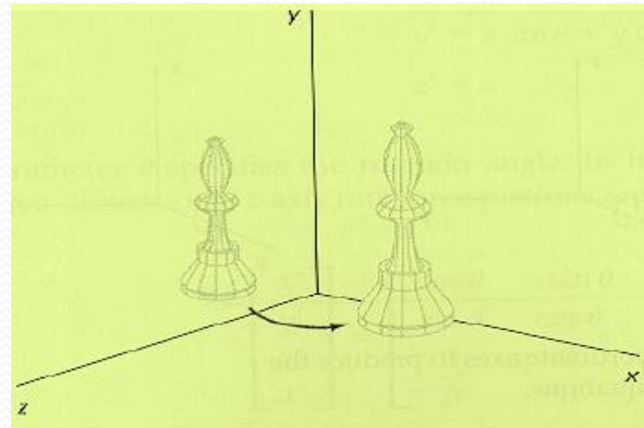
$$(\vec{i}\times\vec{i})\times\vec{j}=0$$

$$\vec{i}\times(\vec{i}\times\vec{j})=\vec{i}\times\vec{k}=-\vec{j}$$

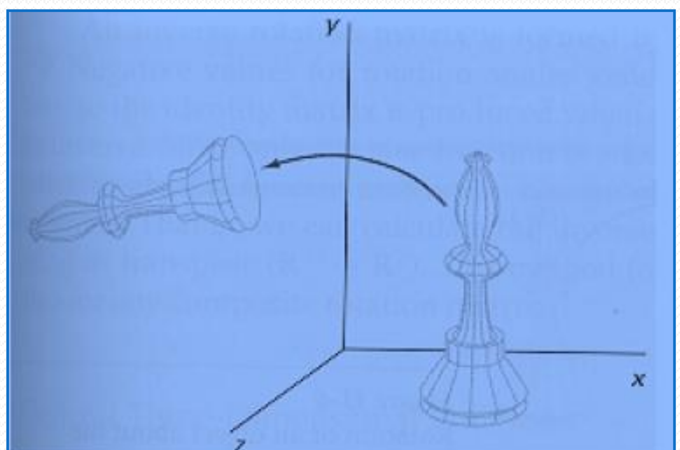
Scalar – Vector – Tensor



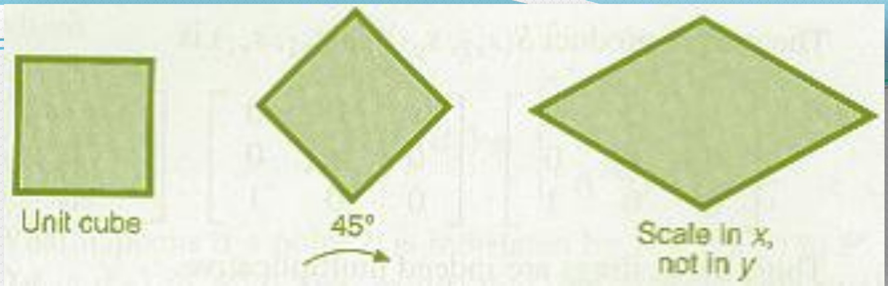
Rotation about the *x*-axis

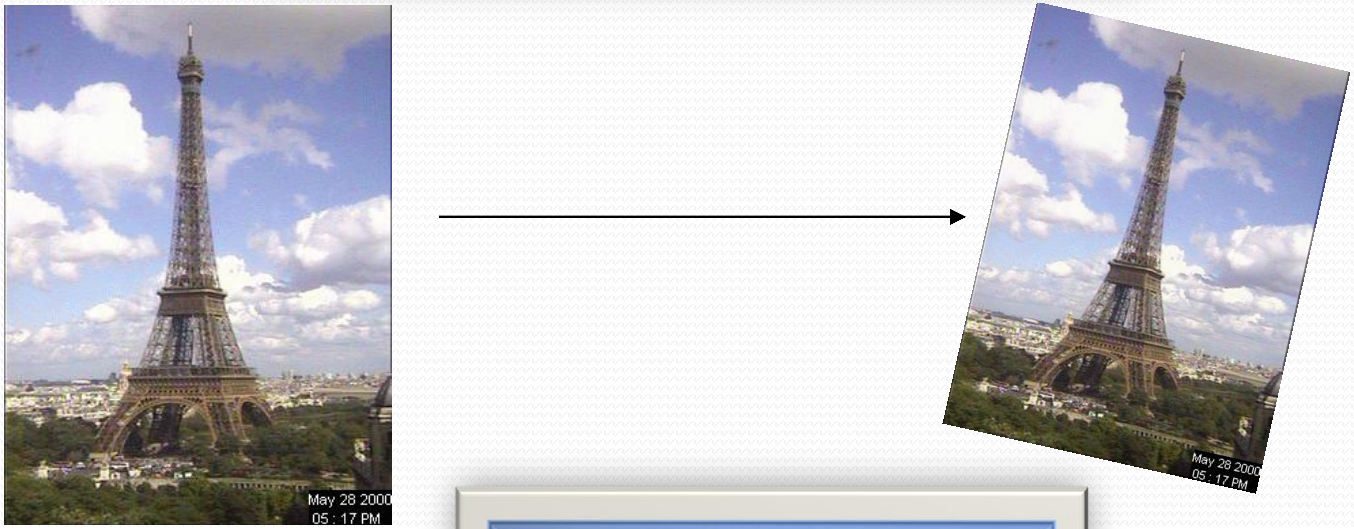


Rotation about the *y*-axis



Rotation about the *z*-axis


Unit cube
45°
Scale in x,
not in y


May 28 2000
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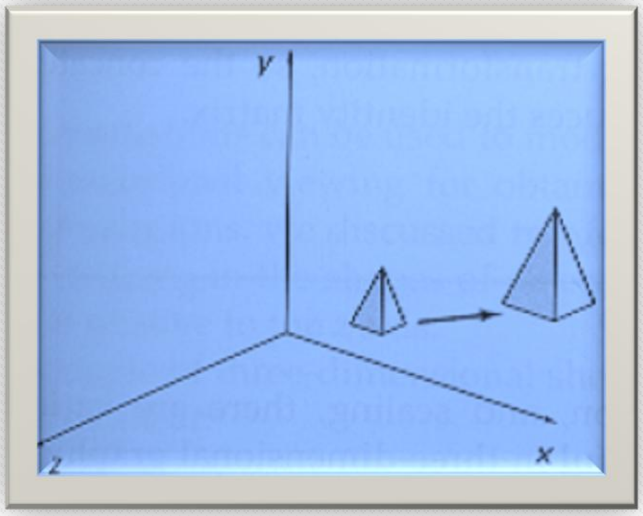
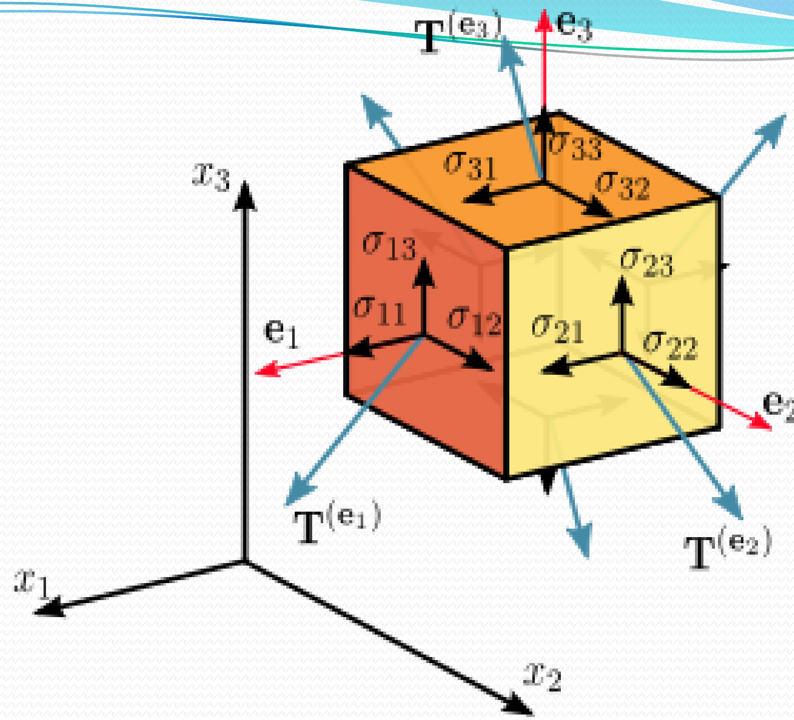
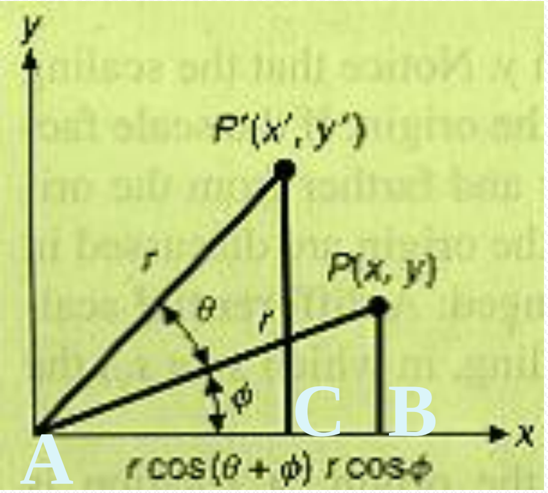

y
z
x



$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

2D Rotation

Rotates points by an angle θ about origin (θ >0: counterclockwise rotation)



From *ABP* triangle:

$$\cos(\phi) = x/r \qquad x = r\cos(\phi)$$
$$\sin(\phi) = y/r \qquad y = r\sin(\phi)$$

From *ACP'* triangle:

$$\cos(\phi+\theta) = x'/r \qquad x' = r\cos(\phi+\theta) = r\cos(\phi)\cos(\theta) - r\sin(\phi)\sin(\theta)$$
$$\sin(\phi+\theta) = y'/r \qquad y' = r\sin(\phi+\theta) = r\cos(\phi)\sin(\theta) + r\sin(\phi)\cos(\theta)$$

$$x' = x\cos(\theta) - y\sin(\theta)$$

$$y' = x\sin(\theta) + y\cos(\theta)$$

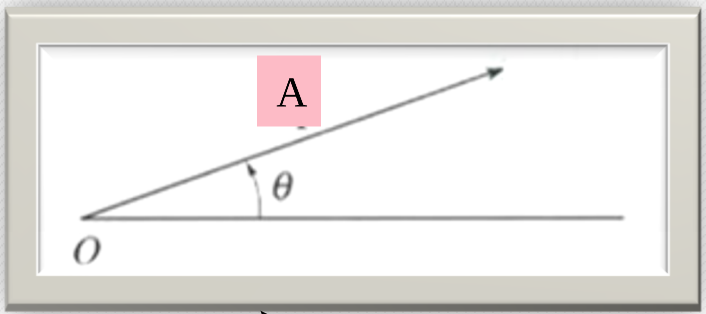
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Vectors

These components are the projections of the vector on the basis vectors of the coordinate system chosen to describe the vectors.

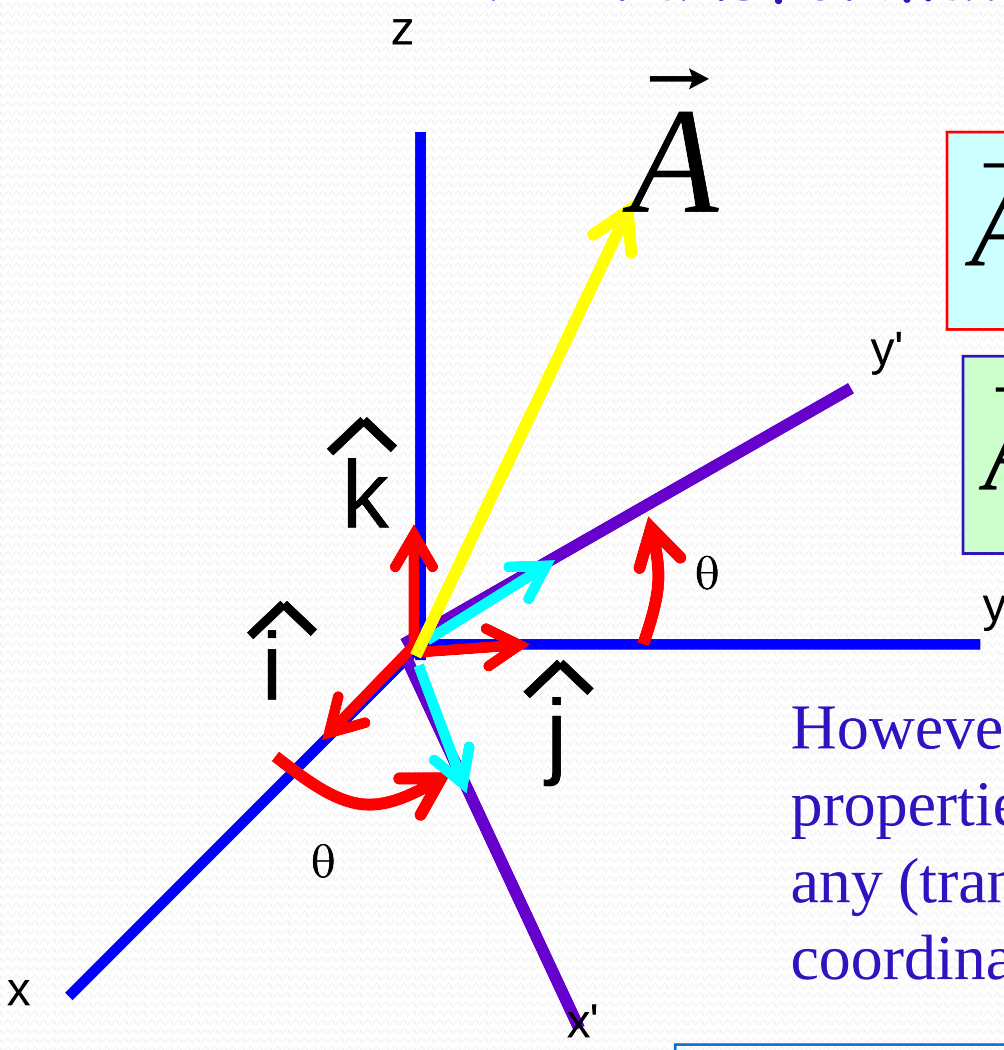
$$A_x = \vec{A}.\hat{i} \quad ; \quad A_y = \vec{A}.\hat{j} \quad ; \quad A_z = \vec{A}.\hat{k}$$

The (numerical) values of the components will, therefore, be different for different choices of basis vectors.



$$A_x = \vec{A}.\hat{i} = A\cos\theta$$

Change of Coordinate System.
The Transformation Matrix



Rotation about the *z*-axis

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\vec{A} = A_{x'}\hat{i}' + A_{y'}\hat{j}' + A_{z'}\hat{k}'$$

However , vectors have two properties that are <u>invariant</u> under any (transformation) change of coordinate axes.

Magnitude	Direction

Vector transformation

$$\vec{\boldsymbol{A}} = \boldsymbol{A}_x \hat{\boldsymbol{i}} + \boldsymbol{A}_y \hat{\boldsymbol{j}} + \boldsymbol{A}_z \hat{\boldsymbol{k}}$$

In the original system of Cartesian coordinates.

$$\vec{A} = A_{x'} \hat{i}' + A_{y'} \hat{j}' + A_{z'} \hat{k}'$$

In the rotated system of Cartesian coordinates.

$$A_{x'} = \vec{A}.\vec{i}' = (\vec{i}.\vec{i}')A_x + (\vec{j}.\vec{i}')A_y + (\vec{k}.\vec{i}')A_z$$

$$A_{y'} = \vec{A}.\vec{j}' = (\vec{i}.\vec{j}')A_x + (\vec{j}.\vec{j}')A_y + (\vec{k}.\vec{j}')A_z$$

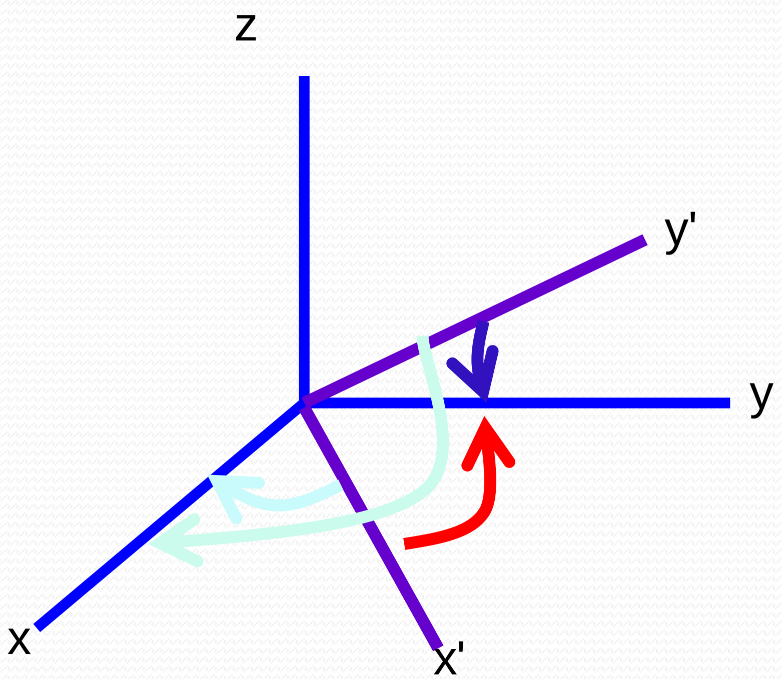
$$A_{z'} = \vec{A}.\vec{k}' = (\vec{i}.\vec{k}')A_x + (\vec{j}.\vec{k}')A_y + (\vec{k}.\vec{k}')A_z$$

$$\begin{pmatrix} \mathbf{A}_{x'} \\ \mathbf{A}_{y'} \\ \mathbf{A}_{z'} \end{pmatrix} = \begin{pmatrix} i.i' & j.i' & k.i' \\ i.j' & j.j' & k.j' \\ i.k' & j.k' & k.k' \end{pmatrix} \begin{pmatrix} \mathbf{A}_{x} \\ \mathbf{A}_{y} \\ \mathbf{A}_{z} \end{pmatrix}$$

$$A_x = \vec{A}.\vec{i} = (\vec{i}'.\vec{i})A_{x'} + (\vec{j}'.\vec{i})A_{y'} + (\vec{k}'.\vec{i})A_{z'}$$

$$A_y = \vec{A}.\vec{j} = (\vec{i}'.\vec{j})A_{x'} + (\vec{j}'.\vec{j})A_{y'} + (\vec{k}'.\vec{j})A_{z'}$$

$$A_z = \vec{A}.\vec{k} = (\vec{i}'.\vec{k})A_{x'} + (\vec{j}'.\vec{k})A_{y'} + (\vec{k}'.\vec{k})A_{z'}$$



$$\hat{i}'.\hat{i} = \cos\theta$$

$$\hat{i}'.\hat{j} = \sin\theta$$

$$\hat{j}'.\hat{i} = -\sin\theta$$

$$\hat{j}'.\hat{j} = \cos\theta$$

$$\hat{k}'.\hat{k} = 1$$

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} i.i' & j.i' & k.i' \\ i.j' & j.j' & k.j' \\ i.k' & j.k' & k.k' \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Example: $\vec{A} = 3\hat{i} + 2\hat{j} + \hat{k}$ $\qquad \sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$A_{x'} = A_x \cos\theta + A_y \sin\theta \quad = \frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{5}{\sqrt{2}}$$

$$A_{y'} = -A_x \sin\theta + A_y \cos\theta \quad = \frac{-3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

$$A_{z'} = A_z = 1$$

Rotation about the *y*-axis

$$\begin{pmatrix} A_{x''} \\ A_{y''} \\ A_{z''} \end{pmatrix} = \begin{pmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{pmatrix} \begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix}$$

Combination of two rotations, the first being about the *z*-axis and the second about the new y’ axis

$$\begin{pmatrix} A_{x''} \\ A_{y''} \\ A_{z''} \end{pmatrix} = \begin{pmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\begin{pmatrix} A_{x''} \\ A_{y''} \\ A_{z''} \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \\ -\sin\theta & \cos\theta & 0 \\ \sin\varphi\cos\theta & \sin\varphi\sin\theta & \cos\varphi \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Derivative of vector

$$\vec{A} = \vec{A}(t) = A_x(t)\vec{i} + A_y(t)\vec{j} + A_z(t)\vec{k}$$

$$\frac{d\vec{A}}{dt} \equiv \lim_{\Delta t \to 0} \frac{\Delta\vec{A}}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{\Delta A_x}{\Delta t}\vec{i} + \frac{\Delta A_y}{\Delta t}\vec{j} + \frac{\Delta A_z}{\Delta t}\vec{k}\right)$$

$$\Delta A_x = A_x(t + \Delta t) - A_x(t)$$

$$\frac{d\vec{A}}{dt} = \frac{dA_x}{dt}\vec{i} + \frac{dA_y}{dt}\vec{j} + \frac{dA_z}{dt}\vec{k}$$

$$\frac{d}{dt}(\vec{A} + \vec{B}) = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$$

$$\frac{d}{dt}(\alpha\mathbf{A} \pm \beta\mathbf{B}) = \alpha\frac{d\mathbf{A}}{dt} \pm \beta\frac{d\mathbf{B}}{dt}$$

Derivative of vector PRODUCT

$$\frac{d(n\vec{A})}{dt} = \lim_{\Delta t \to 0} \frac{n(t+\Delta t)\vec{A}(t+\Delta t) - n(t)\vec{A}(t)}{\Delta t}$$

$$\frac{d(\vec{A}.\vec{B})}{dt} = \lim_{\Delta t \to 0} \frac{\vec{A}(t+\Delta t).\vec{B}(t+\Delta t) - \vec{A}(t).\vec{B}(t)}{\Delta t}$$

$$\frac{d(\vec{A}\times\vec{B})}{dt} = \lim_{\Delta t \to 0} \frac{\vec{A}(t+\Delta t)\times\vec{B}(t+\Delta t) - \vec{A}(t)\times\vec{B}(t)}{\Delta t}$$

$$n(t+\Delta t)\vec{A}(t)$$

$$\frac{d(n\vec{A})}{dt} = \frac{dn}{dt}\vec{A} + n\frac{d\vec{A}}{dt}$$

$$\frac{d(\vec{A}.\vec{B})}{dt} = \frac{d\vec{A}}{dt}.\vec{B} + \vec{A}.\frac{d\vec{B}}{dt}$$

$$\frac{d(\vec{A}\times\vec{B})}{dt} = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}$$

Integral of vector

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$d\vec{F} = (dF_x)\vec{i} + (dF_y)\vec{j} + (dF_z)\vec{k}$$

$$\int \vec{F}.d\vec{r} = \int (F_x dx) + \int (F_y dy) + \int (F_z dy)$$