ANALYTICAL MECHANICS 1

Lecture 19

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Potential Energy in a General Central Field

$$\vec{F} = f(r)\hat{e}_r$$

$$\vec{\nabla} \times \vec{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & \hat{e}_\theta r & \hat{e}_\varphi r\sin\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ F_r & rF_\theta & rF_\varphi \sin\theta \end{vmatrix}$$

$$F_r = f(r) \qquad F_\theta = 0 \qquad F_\varphi = 0$$

$$\vec{\nabla} \times \vec{F} = \frac{\hat{e}_\theta}{r\sin\theta}\frac{\partial f}{\partial \varphi} - \frac{\hat{e}_\varphi}{r}\frac{\partial f}{\partial \theta} = 0$$

$$V(r) = -\int_{r_{ref}}^{r} \vec{F}.d\vec{r} = -\int_{r_{ref}}^{r} f(r)\hat{e}_r.d\vec{r} = -\int_{r_{ref}}^{r} f(r)dr$$

$$\vec{F} = -\vec{\nabla}V \qquad f(r) = -\frac{dV(r)}{dr}$$

Angular Momentum in Central Fields

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \qquad \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

$$\frac{d\vec{L}}{dt} = 0 \qquad \vec{L} = \text{constant vector}$$

Magnitude of the Angular Momentum

$$\vec{V} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\vec{L} = \left|\vec{r} \times m\vec{V}\right| = \left|r\hat{e}_r \times m(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta)\right|$$

$$\left|\hat{e}_r \times \hat{e}_r\right| = 0 \qquad \left|\hat{e}_r \times \hat{e}_\theta\right| = 1$$

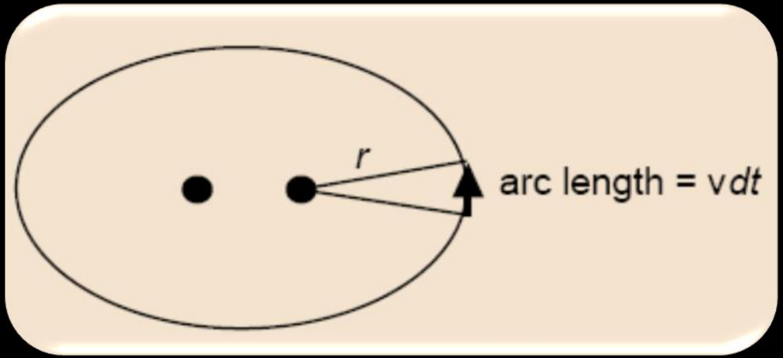
$$\vec{L} = mr^2\dot{\theta}\hat{e}_r \times \hat{e}_\theta = mr^2\dot{\theta}\hat{k}$$

$$\frac{\left|\vec{L}\right|}{m} = r^2\dot{\theta} = h$$

The Law of Areas. Kepler's Laws of Planetary Motion

This observation doesn't require an inverse square force to be valid; it only requires that angular momentum be conserved. Consider a small change in orbital position in time *dt*

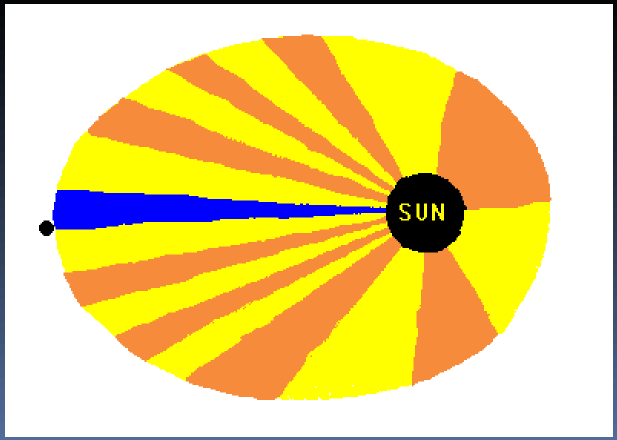
$$dA = \frac{1}{2}\left|\vec{r} \times \vec{v}dt\right| = \frac{1}{2}dt\left|\vec{r} \times \vec{v}\right|$$



$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

$$dA = \frac{1}{2m}\left|\vec{r} \times m\vec{v}\right|dt = \frac{1}{2m}\left|\vec{L}\right|dt = \frac{h}{2}dt$$

$$\frac{dA}{dt} = \frac{h}{2} = cte$$



Kepler's Laws

Kepler's Laws of Planetary Motion

Here is a summary of Kepler's 3 Laws:

1. Planets move around the sun in elliptical paths with the sun at one focus of the ellipse.
2. While orbiting, a planet sweep out equal areas in equal times.
3. The square of a planet's period (revolution time) is proportional to the cube of its mean distance from the sun: $T^2 \propto r^3$

Orbit of a Particle in a Central Force Field

$$m\ddot{\vec{r}} = f(r)\hat{e}_r$$

$$m\ddot{\vec{r}} = m\left[(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta\right] = f(r)\hat{e}_r$$

$$m(\ddot{r} - r\dot{\theta}^2) = f(r)$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \qquad \frac{d}{dt}(r^2\dot{\theta}) = 0$$

$$r^2\dot{\theta} = const\cdot = h$$

$$|h| = \frac{L}{m} = \left|\vec{r} \times \vec{V}\right|$$

$$L = mr^2\dot{\theta}$$

To find equation of the orbit solve the pair of differential equations

$$m(\ddot{r}-r\dot{\theta}^2)=f(r) \qquad m(r\ddot{\theta}+2\dot{r}\dot{\theta})=0$$

$$r=\frac{1}{u} \qquad \dot{r}=-\frac{1}{u^2}\dot{u}=-\frac{1}{u^2}\dot{\theta}\frac{du}{d\theta}=-h\frac{du}{d\theta}$$

$$\dot{\theta}=hu^2$$

$$\ddot{r}=-h\frac{d}{dt}\frac{du}{d\theta}=-h\dot{\theta}\frac{d^2u}{d\theta^2}=-h^2u^2\frac{d^2u}{d\theta^2}$$

$$\frac{d^2u}{d\theta^2}+u=-\frac{1}{mh^2u^2}f(u^{-1})$$

$$Example\,3:$$

$$r = c\theta^2 \qquad u = \frac{1}{c\theta^2}$$

$$\frac{du}{d\theta} = \frac{-2}{c}\theta^{-3} \qquad \frac{d^2u}{d\theta^2} = \frac{6}{c}\theta^{-4} = \frac{6}{c}c^2u^2 = 6cu^2$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{mh^2u^2}f(u^{-1})$$

$$6cu^2 + u = -\frac{1}{mh^2u^2}f(u^{-1})$$

$$f(u^{-1}) = -mh^2(6cu^4 + u^3)$$

$$f(r) = -mh^2\left(\frac{6c}{r^4} + \frac{1}{r^3}\right)$$

$Example\,4:$

$$\theta(t) = ?$$

$$h = r^2\dot{\theta}$$

$$\dot{\theta} = hu^2 = h\frac{1}{c^2\theta^4}$$

$$\int \theta^4 d\theta = \int \frac{h}{c^2}dt$$

$$\frac{\theta^5}{5} = hc^{-2}t + c_1$$

$$t = 0, \theta = 0 \rightarrow c_1 = 0$$

$$\theta = (5hc^{-2})^{1/5} t^{1/5}$$

$$\alpha = (5hc^{-2})^{1/5} = const \cdot$$

$$\theta = \alpha t^{1/5}$$

Energy Equation of the Orbit

$$\frac{1}{2}mv^2 + V(r) = E \qquad v^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E \qquad \dot{\theta} = hu^2$$

$$r = \frac{1}{u} \qquad \dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{1}{u^2}\dot{\theta}\frac{du}{d\theta} = -h\frac{du}{d\theta}$$

$$\frac{1}{2}m[h^2(\frac{du}{d\theta})^2 + \frac{1}{u^2}h^2u^4] + V(u^{-1}) = E$$

$$\frac{1}{2}mh^2\left[(\frac{du}{d\theta})^2 + u^2\right] + V(u^{-1}) = E$$

Example 5:

$$r = c\theta^2 \qquad u = \frac{1}{r} = \frac{1}{c}\theta^{-2} \qquad \theta^{-2} = cu$$

$$\frac{du}{d\theta} = \frac{-2}{c}\theta^{-3} = -2c^{1/2}u^{3/2}$$

$$\frac{1}{2}mh^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] + V(u^{-1}) = E$$

$$\frac{1}{2}mh^2(4cu^3 + u^2) + V(u^{-1}) = E$$

$$V(r) = E - \frac{1}{2} m h^2 \left(\frac{4c}{r^3} + \frac{1}{r^2} \right)$$

$$f(r) = -dV/dr$$

$$f(r) = \frac{1}{2} m h^2 \left(\frac{-12cr^2}{r^6} - \frac{2r}{r^4} \right)$$

$$f(r) = -\frac{6mch^2}{r^4} - \frac{mh^2}{r^3}$$

Orbits in an Inverse-Square Field

$$f(r) = -\frac{k}{r^2}$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{mh^2u^2} f(u^{-1})$$

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{mh^2}$$

$$u_g = A\cos(\theta - \theta_\circ)$$

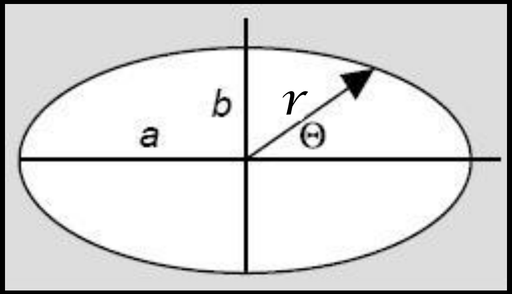
$$\theta_\circ = 0 \rightarrow u_g = A\cos\theta$$

$$u_p = \frac{k}{mh^2}$$

$$u = A\cos\theta + \frac{k}{mh^2}$$

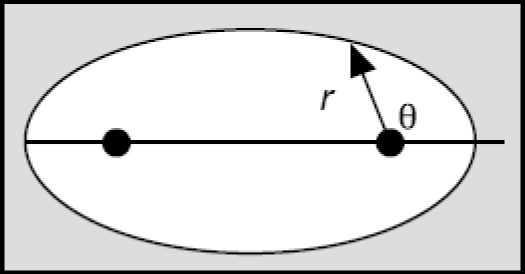
$$r = \frac{1}{A\cos\theta + k/mh^2}$$

$$e^2 = 1 - \frac{b^2}{a^2}, \quad a > b$$



$$e < 1 \rightarrow ellipse$$

$$e = 0 \rightarrow circle$$



$$e = 1 \rightarrow parabola$$

$$e > 1 \rightarrow hyperbola$$

$$r = \frac{mh^2/k}{1+\frac{Amh^2}{k}\cos\theta} \quad \overset{compare}{\leftrightarrow} \quad r = r_\circ \frac{1+e}{1+e\cos\theta}$$

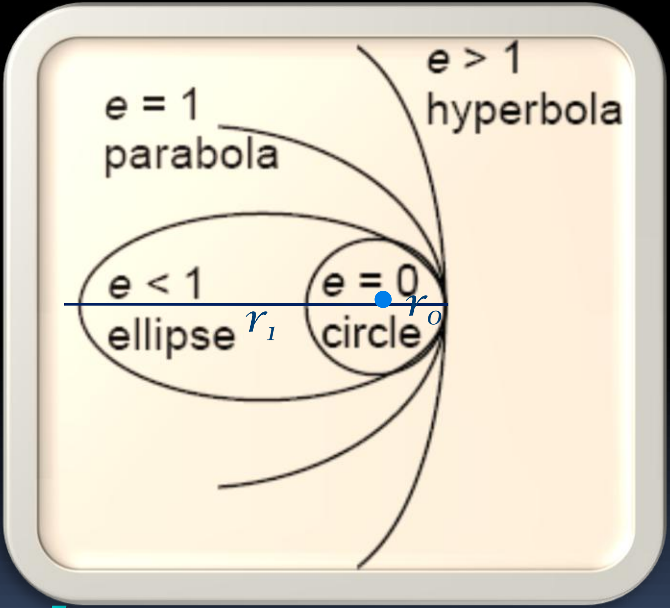
$$e = \frac{Amh^2}{k} \qquad r_\circ = \frac{mh^2}{k(1+e)} \rightarrow e = \frac{mh^2}{kr_\circ} - 1$$

$$e < 1 \rightarrow ellipse$$

$$e = 0 \rightarrow circle$$

$$e = 1 \rightarrow parabola$$

$$e > 1 \rightarrow hyperbola$$



$$if\ \theta = 0 \rightarrow r = r_\circ$$

$$if\ \theta = \pi \rightarrow r_1 = r_\circ \frac{1+e}{1-e} = \frac{mh^2/k}{1-e} \quad if\ 0 < \theta < \pi \rightarrow r_\circ < r < r_1$$