ANALYTICAL MECHANICS 1

Lecture 18

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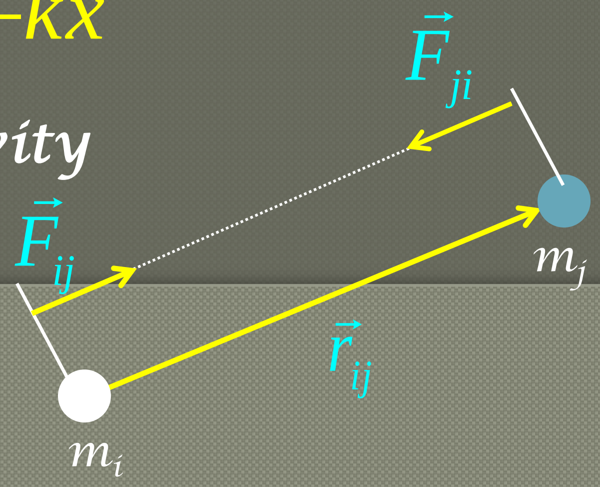
Central Forces and Celestial Mechanics

$$Hook\text{`}s\ Law\quad \vec{F} = -k\vec{x}$$

Newton's Law of Gravity



$$\vec{F}_{ij} = G\frac{m_i m_j}{r_{ij}^{\ 2}}\left(\frac{\vec{r}_{ij}}{r_{ij}}\right)$$

$$G = 6.67\times10^{-11}\,Nm^2/kg^2$$

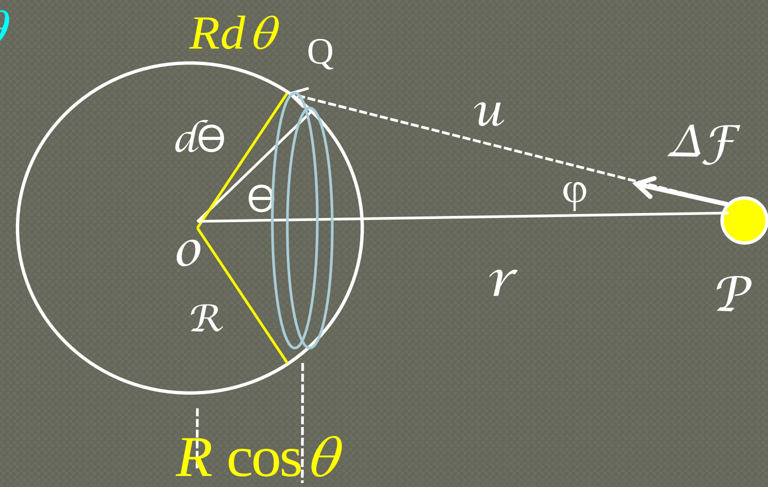
$$\vec{F}_{ij} = -\vec{F}_{ji}$$

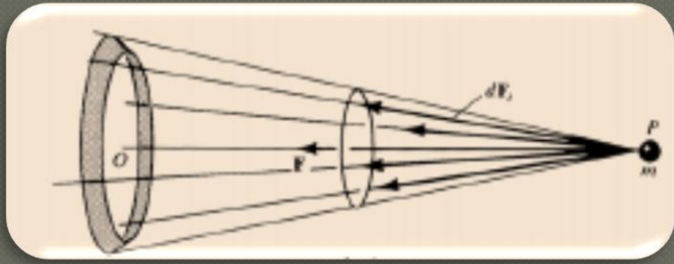
$$Electrostatics\quad \vec{F}_{ij} = k\frac{q_i q_j}{r_{ij}^{\ 2}}\left(\frac{\vec{r}_{ij}}{r_{ij}}\right)$$

$$k = 8.99\times10^{9}\,Nm^2/C^2$$

Gravitational Forces Between a Uniform Sphere and a Particle

The circumference of the ring is $2\pi R\sin\theta$





$$ds = 2\pi R\sin\theta R\Delta\theta$$

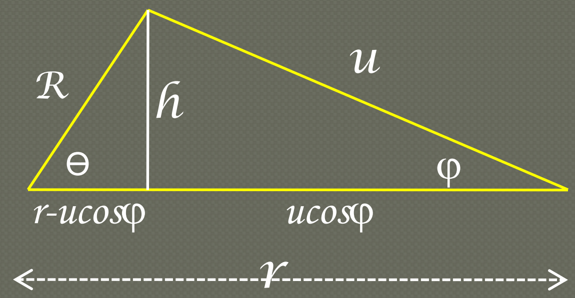
$$dM = \rho 2\pi R\sin\theta R d\theta$$

$$F = \int F\cos\varphi$$

$$F = \int G\frac{m dM}{u^2}\cos\varphi$$

$$= Gm2\pi\rho R^2\int_0^{\pi}\frac{\sin\theta\cos\varphi d\theta}{u^2}$$

$$h^2 = R^2 - (r - u\cos\varphi)^2$$



$$h^2 = u^2 - (u\cos\varphi)^2$$

$$R^2-(r^2+u^2\cos^2\varphi-2ru\cos\varphi)=u^2-u^2\cos^2\varphi$$

$$R^2=u^2+r^2-2ru\cos\varphi$$

Also from the law of cosines

$$u^2=r^2+R^2-2rR\cos\theta$$

$$2udu=2rR\sin\theta d\theta$$

$$\cos\varphi=\frac{u^2+r^2-R^2}{2ru}$$

$$F=Gm2\pi\rho R^2\int_{\theta=0}^{\theta=\pi}\frac{u^2+r^2-R^2}{2Rr^2u^2}du$$

$$M = \rho 4\pi R^2$$

$$F = \frac{GmM}{4Rr^2}\int_{r-R}^{r+R}\left(1+\frac{r^2-R^2}{u^2}\right)du$$

$$F = \frac{GmM}{4Rr^2}\left[u - \frac{r^2}{u} + \frac{R^2}{u}\right]_{r-R}^{r+R}$$

$$F = \frac{GmM}{r^2}$$

$$\vec{F} = -G\frac{Mm}{r^2}\hat{e}_r$$

Potential Energy in a Gravitational Field.

Gravitational Potential

$$\vec{F} = -\frac{GMm}{r^2}\hat{e}_r$$

$$dW = -\vec{F}.d\vec{r} = \frac{GMm}{r^2}\hat{e}_r.d\vec{r}$$

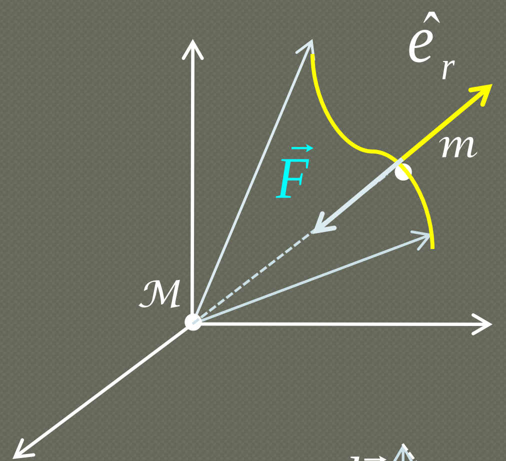
$$\hat{e}_r.d\vec{r} = dr$$

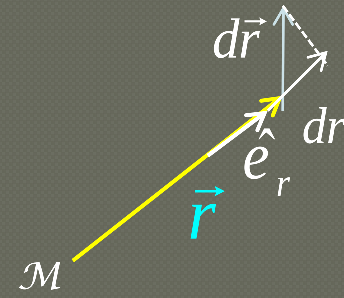
$$W = GMm\int_{r_1}^{r_2}\frac{dr}{r^2} = -GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

$$V(r) = GMm\int_{\infty}^{r}\frac{dr}{r^2} = -\frac{GMm}{r}$$

$$\Phi = \frac{V}{m}$$ *gravitational potential*

$$\Phi = -\frac{GM}{r}$$





$$\Phi(x,y,z)=\sum \Phi_i=-G\sum \frac{M_i}{u_i}$$

$$u_i=\left|\vec{r}-\vec{r}_i\right|$$

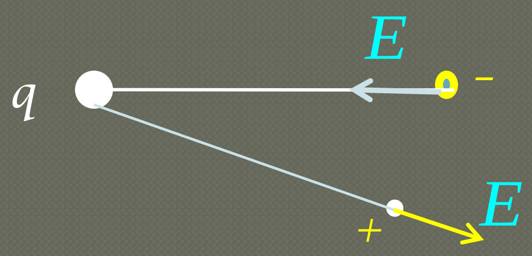
gravitational field intensity

$$F=\frac{GmM}{r^2}$$

$$\vec{g}^*=\frac{\vec{F}}{m}=G\frac{M}{r^2}$$

$$F=\frac{kqq_\circ}{r^2}$$

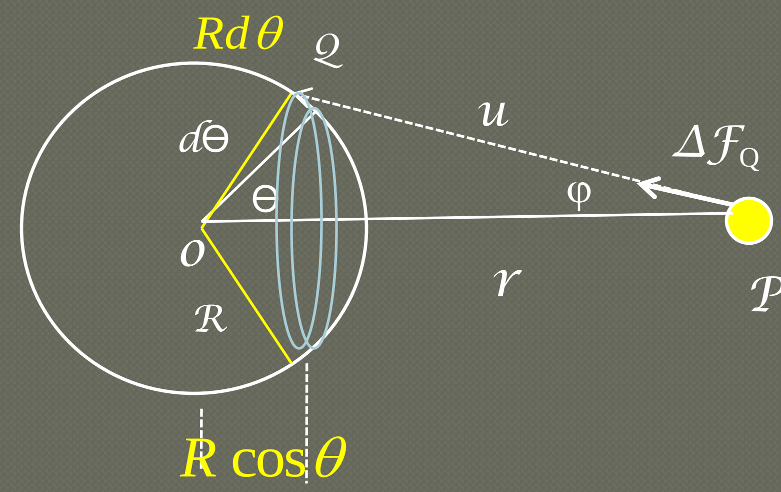
$$E=\frac{F}{q_\circ}=\frac{kq}{r^2}$$



$$\vec{g}^*=-\nabla\Phi$$

$$\vec{F}=-\vec{\nabla}V$$

Example1:



$$\Phi = -G\int \frac{dM}{u}$$

$$= -G\int \frac{2\pi\rho R^2 \sin\theta d\theta}{u}$$

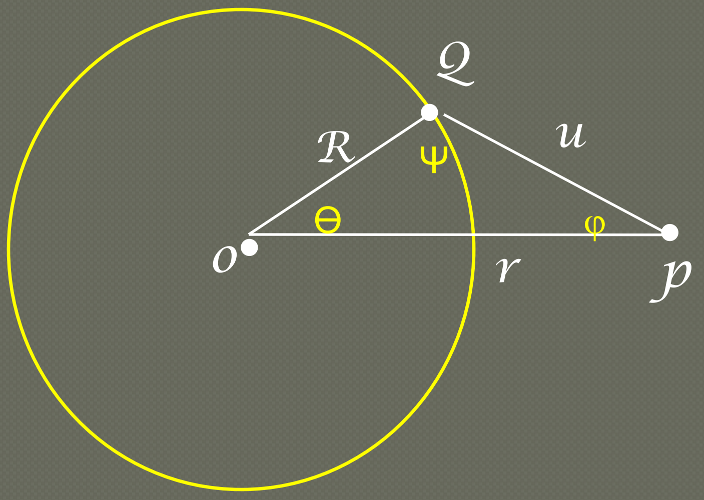
$$u^2 = r^2 + R^2 - 2rR\cos\theta$$

$$2udu = 2rR\sin\theta d\theta$$

$$\Phi = -G\frac{2\pi\rho R^2}{rR}\int_{r-R}^{r+R} du = -G\frac{4\pi\rho R^2}{r} = -\frac{GM}{r}$$

Example 2:

$$\Phi = -G\int \frac{dM}{u} = -G\int_0^{2\pi} \frac{\mu R d\theta}{u}$$



$$\frac{\sin\psi}{r} = \frac{\sin\varphi}{R}$$

$$R\sin\psi = r\sin\varphi$$

$$R\cos\psi d\psi = r\cos\varphi d\varphi = r\cos\varphi(-d\theta - d\psi)$$

$$\theta + \varphi + \psi = \pi$$

$$u = R\cos\psi + r\cos\varphi$$

$$u d\psi = -r\cos\varphi d\theta = -(r^2 - R^2\sin^2\psi)^{1/2} d\theta$$

$$\Phi = -G\mu R\, 4\int_0^{\pi/2} (r^2 - R^2 \sin^2\psi)^{-1/2} d\psi$$

$$\Phi = -G\frac{4\mu R}{r}\int_0^{\pi/2} (1 + \frac{1}{2}\frac{R^2}{r^2}\sin^2\psi + \cdots) d\psi$$

$$= -G\frac{4\mu R}{r}(\frac{\pi}{2} + \frac{\pi R^2}{8r^2} + \cdots)$$

$$= -\frac{GM}{r}(1 + \frac{R^2}{4r^2} + \cdots)$$

$$\vec{g}^{\,*} = -\frac{\partial\Phi}{\partial r}\hat{e}_r = (-\frac{GM}{r^2} - \frac{3GMR^2}{4r^4} - \cdots)\hat{e}_r$$