ANALYTICAL MECHANICS 1

Lecture 17

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Dynamic Effects. Motion of a Projectile

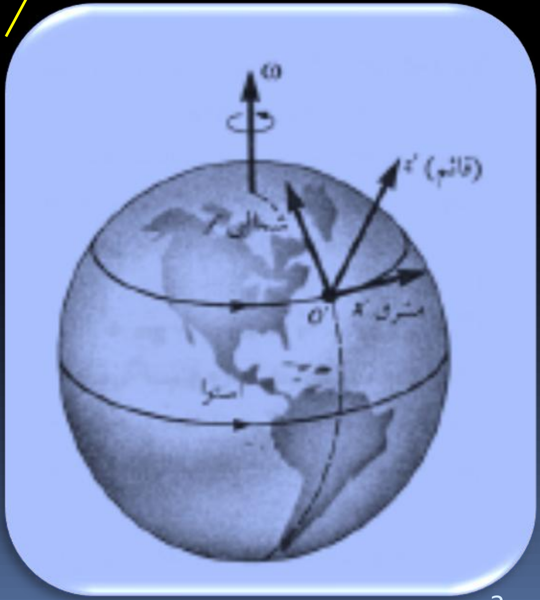
$$\vec{F} - m\vec{A}_0 - 2m\vec{\omega}\times\vec{v}' - m\dot{\vec{\omega}}\times\vec{r}' - m\vec{\omega}\times(\vec{\omega}\times\vec{r}') = m\vec{a}'$$

$$m\ddot{\vec{r}}' = \vec{F} + m\vec{g}_* - m\vec{A}_0 - 2m\vec{\omega}\times\dot{\vec{r}}' - m\vec{\omega}\times(\vec{\omega}\times\vec{r}')$$

$$m\ddot{\vec{r}}' = \vec{F} + m\vec{g} - 2m\vec{\omega}\times\dot{\vec{r}}' - m\vec{\omega}\times(\vec{\omega}\times\vec{r}')$$

$$m\ddot{\vec{r}}' = m\vec{g} - 2m\vec{\omega}\times\dot{\vec{r}}'$$

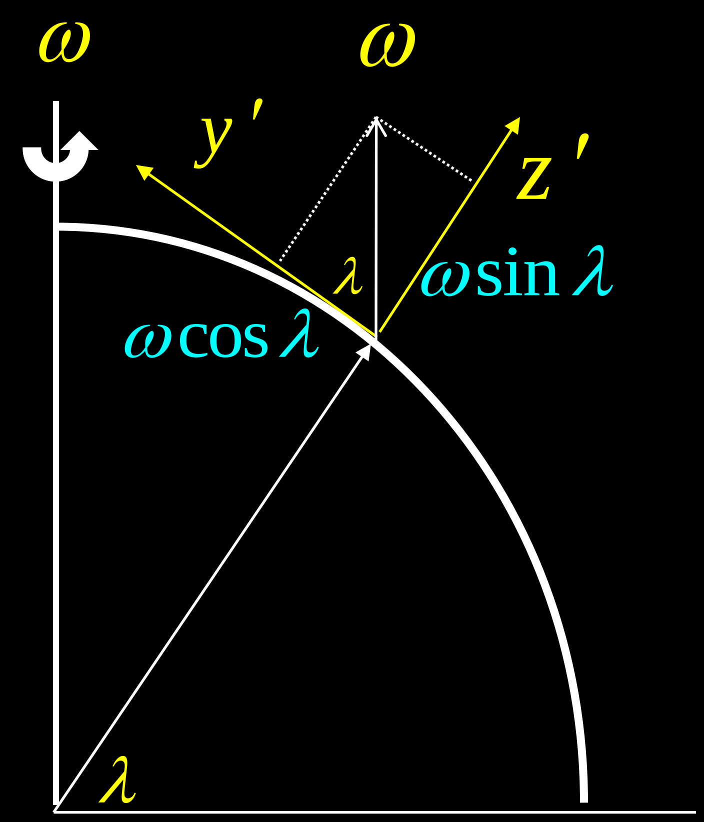
$$\vec{g} = -g\hat{k}'$$



$$\omega_{x'} = 0$$

$$\omega_{y'} = \omega\cos\lambda$$

$$\omega_{z'} = \omega\sin\lambda$$



$$\vec{\omega}\times\dot{\vec{r}}' = \begin{vmatrix} \hat{i}' & \hat{j}' & \hat{k}' \\ 0 & \omega\cos\lambda & \omega\sin\lambda \\ \dot{x}' & \dot{y}' & \dot{z}' \end{vmatrix}$$

$$= \hat{i}'(\omega\dot{z}'\cos\lambda - \omega\dot{y}'\sin\lambda) + \hat{j}'(\omega\dot{x}'\sin\lambda) + \hat{k}'(-\omega\dot{x}'\cos\lambda)$$

$$m\ddot{\vec{r}}' = m\vec{g} - 2m\vec{\omega}\times\dot{\vec{r}}'$$

$$\begin{cases} \ddot{x}' = -2\omega(\dot{z}'\cos\lambda - \dot{y}'\sin\lambda) \\ \ddot{y}' = -2\omega(\dot{x}'\sin\lambda) \\ \ddot{z}' = -g + 2\omega\dot{x}'\cos\lambda \end{cases}$$

$$\begin{cases} \dot{x}' = -2\omega(z'\cos\lambda - y'\sin\lambda) + \dot{x}'_{\circ} \\ \dot{y}' = -2\omega x'\sin\lambda + \dot{y}'_{\circ} \\ \dot{z}' = -gt + 2\omega x'\cos\lambda + \dot{z}'_{\circ} \end{cases}$$

$$\ddot{x}' = 2\omega gt\cos\lambda - 2\omega(\dot{z}'_{\circ}\cos\lambda - \dot{y}'_{\circ}\sin\lambda)$$ Ignore ω^2

$$\dot{x}' = \omega gt^2\cos\lambda - 2\omega t(\dot{z}'_{\circ}\cos\lambda - \dot{y}'_{\circ}\sin\lambda) + \dot{x}'_{\circ}$$

$$x'(t) = \frac{1}{3}\omega g t^3 \cos\lambda - \omega t^2 (\dot{z}'_\circ \cos\lambda - \dot{y}'_\circ \sin\lambda) + \dot{x}'_\circ t + x'_\circ$$

$$\dot{y}' = -2\omega x' \sin\lambda + \dot{y}'_\circ = -2\omega(\dot{x}'_\circ t + x'_\circ)\sin\lambda + \dot{y}'_\circ$$

Ignore ω^2

$$y'(t) = -\omega \dot{x}'_\circ t^2 \sin\lambda - 2\omega x'_\circ t \sin\lambda + \dot{y}'_\circ t + y'_\circ$$

$$\dot{z}' = -gt + 2\omega x' \cos\lambda + \dot{z}'_\circ = -gt + 2\omega(\dot{x}'_\circ t + x'_\circ)\cos\lambda + \dot{z}'_\circ$$

$$z'(t) = -\frac{1}{2}g t^2 + \omega \dot{x}'_\circ t^2 \cos\lambda + 2\omega x'_\circ t \cos\lambda + \dot{z}'_\circ t + z'_\circ$$

$$
\text{if } \omega = 0 \rightarrow \begin{cases} x'(t) = \dot{x}'_{\circ} t + x'_{\circ} \\ y'(t) = \dot{y}'_{\circ} t + y'_{\circ} \\ z'(t) = -\dfrac{1}{2} g t^2 + \dot{z}'_{\circ} t + z'_{\circ} \end{cases}
$$

 *Example 1:*

$$t=0\rightarrow \dot{x}'_{\circ}=\dot{y}'_{\circ}=\dot{z}'_{\circ}=0 \qquad x'_{\circ}=y'_{\circ}=0,\quad z'_{\circ}=h$$

$$\begin{cases} x'(t)=\dfrac{1}{3}\omega g t^{3}\cos\lambda-\omega t^{2}(\dot{z}'_{\circ}\cos\lambda-\dot{y}'_{\circ}\sin\lambda)+\dot{x}'_{\circ}t+x'_{\circ} \\ y'(t)=-\omega\dot{x}'_{\circ}t^{2}\sin\lambda-2\omega x'_{\circ}t\sin\lambda+\dot{y}'_{\circ}t+y'_{\circ} \\ z'(t)=-\dfrac{1}{2}gt^{2}+\omega\dot{x}'_{\circ}t^{2}\cos\lambda+2\omega x'_{\circ}t\cos\lambda+\dot{z}'_{\circ}t+z'_{\circ} \end{cases}$$

$$\begin{cases} x'(t)=\dfrac{1}{3}\omega g t^{3}\cos\lambda \\ y'(t)=0 \\ z'(t)=-\dfrac{1}{2}gt^{2}+h \end{cases}$$

$$z'=0 \qquad t^{2}=2h/g$$

$$x'_{h}=\frac{1}{3}\omega g\left(\frac{2h}{g}\right)^{3/2}\cos\lambda$$

Example 2:

$$h = 100m, \quad \lambda = 45^\circ$$

$$x'_h = \frac{1}{3}\omega g \left(\frac{2h}{g}\right)^{3/2} \cos\lambda$$

$$x'_h = \frac{1}{3}(7.27\times10^{-5}s^{-1})(8\times100^3\, m^3 / 9.8ms^{-2})^{1/2} \cos 45^\circ$$

$$= 1.55\times10^{-2}\, m = 1.55cm \quad \textit{eastward}$$

Example 3:

$$\dot{x}'_{\circ} = v_{\circ}, \quad \dot{y}'_{\circ} = \dot{z}'_{\circ} = 0$$

$$if \quad t = 0, \; x'_{\circ} = y'_{\circ} = z'_{\circ} = 0$$

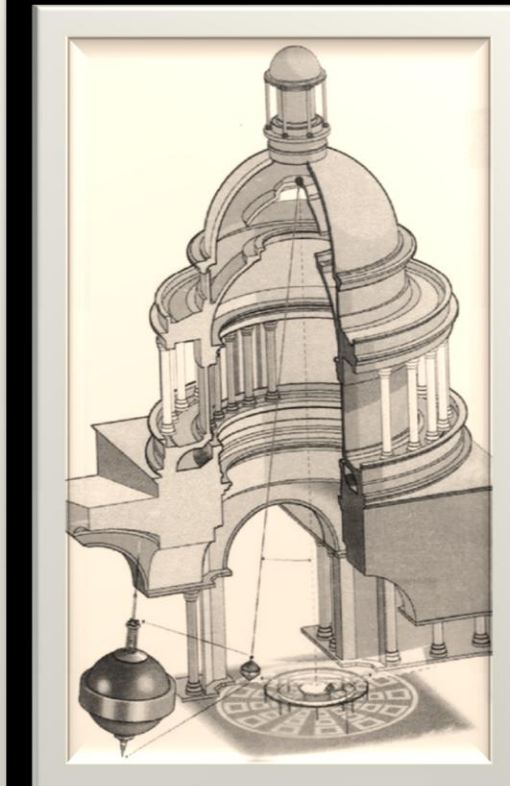
$$y'(t) = -\omega \dot{x}'_{\circ} t^{2} \sin\lambda - 2\omega x'_{\circ} t \sin\lambda + \dot{y}'_{\circ} t + y'_{\circ}$$

$$y'(t) = -\omega v_{\circ} t^{2} \sin\lambda$$

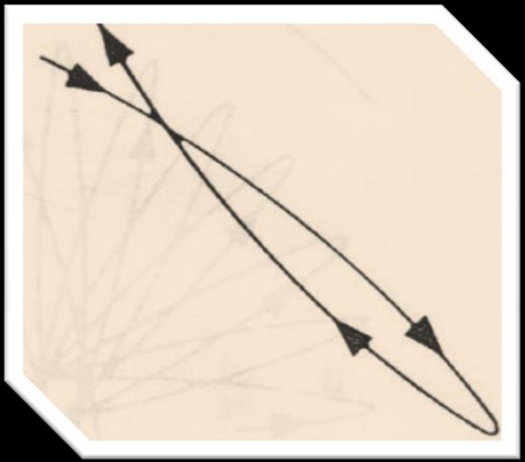
$if \;\; \lambda > 0 \;\; (in \; the \; NH) \rightarrow$ *projectile veers to the south or right*

$if \;\; \lambda < 0 \;\; (in \; the \; SH) \rightarrow$ *projectile veers to the north or left*

The Foucault Pendulum



In essence, the Foucault Pendulum is a Pendulum with a long enough damping rate such that the precession of its plane of oscillations can be observed after typically an hour or more. A whole revolution of the plane of oscillation takes anywhere between a day if it is at the pole, or longer at lower latitudes.

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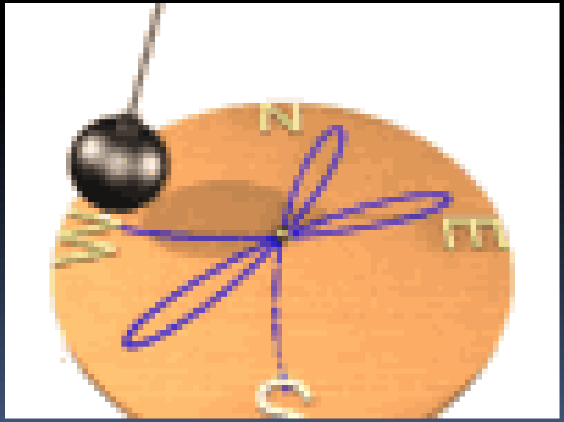
N
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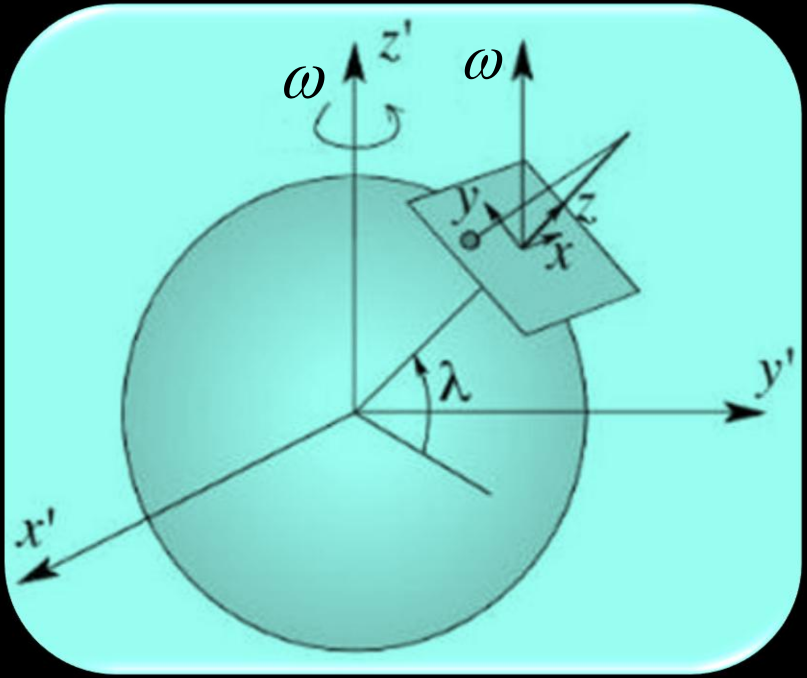
E
C

The Foucault Pendulum



The Rotating coordinate system {x,y,z} is non-inertial since Earth is rotating. As a result, a Coriolis force is added when working in this frame of reference.

$$\vec{F} - m\vec{A}_0 - 2m\vec{\omega}\times\vec{v}' - m\dot{\vec{\omega}}\times\vec{r}' - m\vec{\omega}\times(\vec{\omega}\times\vec{r}') = m\vec{a}'$$

$$\vec{g} = \vec{g}_* - \vec{A}_\circ \qquad m\ddot{\vec{r}}' = m\vec{g} + \vec{S} - 2m\vec{\omega}\times\dot{\vec{r}}'$$

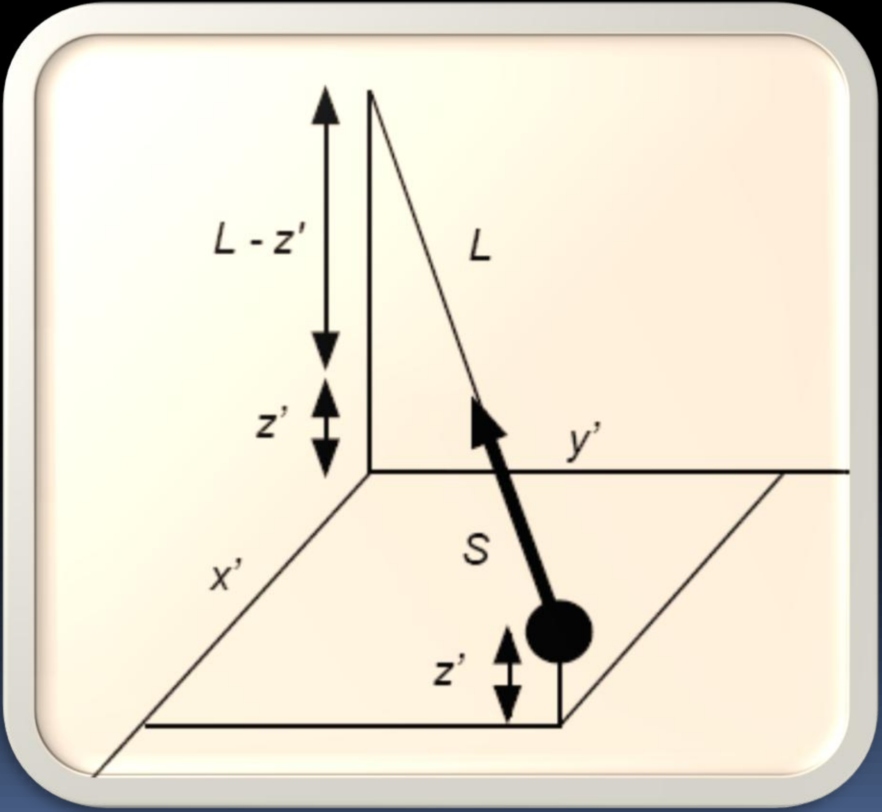
$$\begin{cases} \ddot{x}' = -2\omega(\dot{z}'\cos\lambda - \dot{y}'\sin\lambda) \\ \ddot{y}' = -2\omega(\dot{x}'\sin\lambda) \\ \ddot{z}' = -g + 2\omega\dot{x}'\cos\lambda \end{cases}$$

$$S_{x'}/S = x'/L$$

$$\rightarrow S_{x'} = -(x'/L)S$$

$$S_{y'} = -(y'/L)S$$

$$S_{z'} = \left[(L - z')/L\right]S$$



$$m\ddot{\vec{r}}' = m\vec{g} + \vec{S} - 2m\vec{\omega}\times\dot{\vec{r}}'$$

$$\begin{cases} m\ddot{x}' = -(x'S/L) - 2m\omega(\dot{z}'\cos\lambda - \dot{y}'\sin\lambda) \\ m\ddot{y}' = -(y'S/L) - 2m\omega(\dot{x}'\sin\lambda) \end{cases}$$

$\dot{z}' \simeq 0$ *corresponding to no vertical motion*

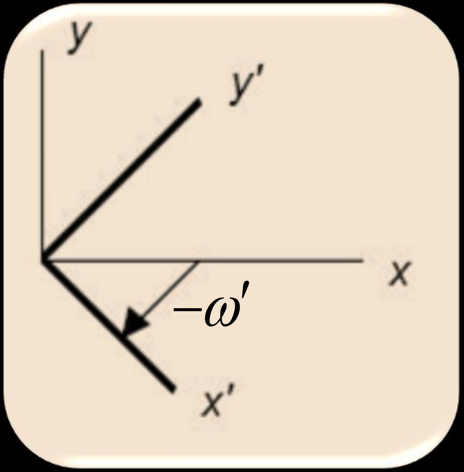
S=mg (in magnitude) since the pendulum is almost vertical

$$\begin{cases} \ddot{x}' = -(g/L)x' + 2\omega'(\dot{y}') \\ \ddot{y}' = -(g/L)y' - 2\omega'(\dot{x}') \end{cases}$$

$$\omega' = \omega\sin\lambda = \omega_{z'}$$

Where ω' is the component of ω in the z' direction (the local vertical component of ω)

$$\begin{cases} x' = x\cos\omega' t + y\sin\omega' t \\ y' = -x\sin\omega' t + y\cos\omega' t \end{cases}$$



$$-\omega' = -\omega\sin\lambda$$

$$\begin{cases} \dot{x}' = \dot{x}\cos\omega' t - \omega' x\sin\omega' t + \dot{y}\sin\omega' t + \omega' y\cos\omega' t \\ \dot{y}' = -\dot{x}\sin\omega' t - \omega' x\cos\omega' t + \dot{y}\cos\omega' t - \omega' y\sin\omega' t \end{cases}$$

$$\begin{cases} \dot{x}' = \dot{x}\cos\omega' t + \dot{y}\sin\omega' t + \omega' y' \\ \dot{y}' = -\dot{x}\sin\omega' t + \dot{y}\cos\omega' t - \omega' x' \end{cases}$$

$$\begin{cases} \ddot{x}' = \ddot{x}\cos\omega' t + \ddot{y}\sin\omega' t + 2\omega'\dot{y}' + \omega'^2 x' \\ \ddot{y}' = -\ddot{x}\sin\omega' t + \ddot{y}\cos\omega' t - 2\omega'\dot{x}' + \omega'^2 y' \end{cases}$$

$$\begin{cases} \ddot{x}' = -(g/L)x' + 2\omega'\dot{y}' \\ \ddot{y}' = -(g/L)y' - 2\omega'\dot{x}' \end{cases}$$

$$\ddot{x}\cos\omega' t + \ddot{y}\sin\omega' t + 2\omega'\dot{y}' + \omega'^2 x' = -\frac{g}{L}x\cos\omega' t$$

$$-\frac{g}{L}y\sin\omega' t + 2\omega'\dot{y}'$$

$$-\ddot{x}\sin\omega' t + \ddot{y}\cos\omega' t - 2\omega'\dot{x}' + \omega'^2 y' = \frac{g}{L}x\sin\omega' t$$

$$-\frac{g}{L}y\cos\omega' t - 2\omega'\dot{x}'$$

$$(\ddot{x} + \frac{g}{L}x)\cos\omega' t + (\ddot{y} + \frac{g}{L}y)\sin\omega' t = 0$$

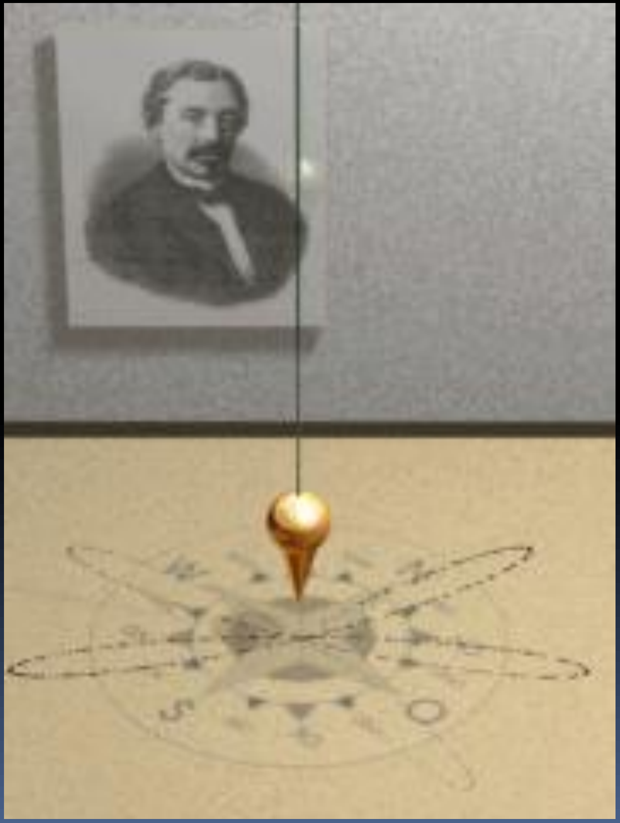
$$-(\ddot{x} + \frac{g}{L}x)\sin\omega' t + (\ddot{y} + \frac{g}{L}y)\cos\omega' t = 0$$

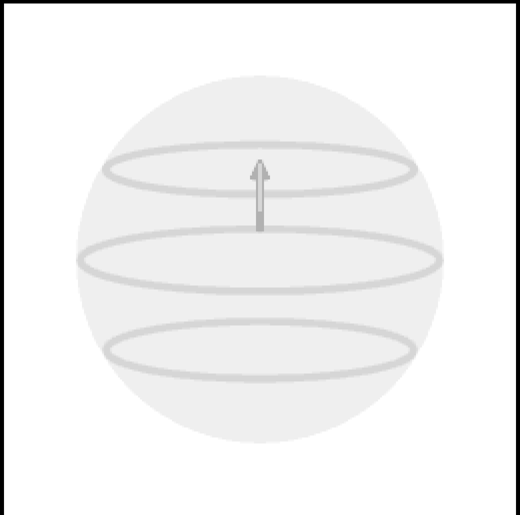
$$\ddot{x} + \frac{g}{L}x = 0 \qquad \ddot{y} + \frac{g}{L}y = 0 \qquad T = 2\pi\sqrt{L/g}$$

What is important to note is that the xy coordinates rotate with an angular frequency ωsinλ=ω' *with respect to x'y' : that is, the plane of oscillation rotates with respect to a coordinate system on the Eath`s surface*

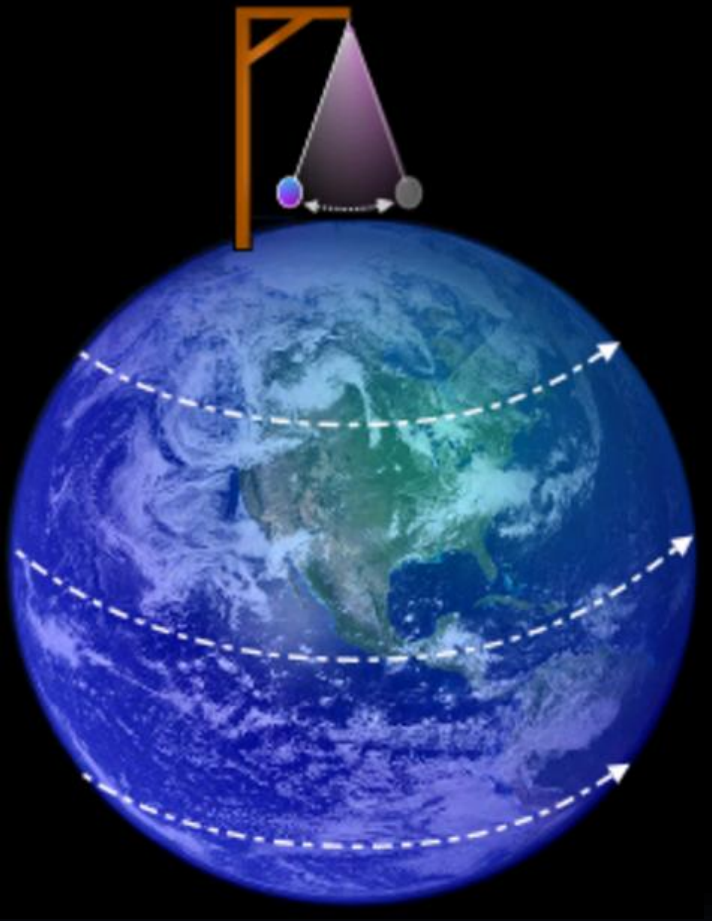
$$T_f = \frac{2\pi}{\omega'} = \frac{2\pi}{\omega \sin\lambda} = \frac{T_e}{\sin\lambda} = \frac{24h}{\sin\lambda} \qquad \begin{cases} \lambda = \pi/2 \rightarrow T' = 24h \\ \lambda = 45^\circ \rightarrow T_f = 34h \\ \lambda = 0 \rightarrow T_f = \infty \end{cases}$$

Away from the equator the co-rotating with the Earth is diminished. Between the poles and the equator the plane of oscillation is rotating both with respect to the stars and with respect to the Earth. The direction of the plane of oscillation of a pendulum with respect to the Earth rotates with an angular speed proportional to the sine of its latitude; thus one at 45° rotates once every 1.4 days and one at 30° every 2 days.





The animation describes the motion of a Foucault Pendulum at a latitude of 30°N. The plane of oscillation rotates by an angle of -180° during one day, so after two days the plane returns to its original orientation.



A Foucault pendulum at the north pole. The pendulum swings in the same plane as the Earth rotates beneath it.

At either the When a Foucault pendulum is suspended somewhere on the equator, then the plane of oscillation of the Foucault pendulum is at all times co-rotating with the rotation of the Earth.