ANALYTICAL MECHANICS 1

Lecture 15

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Chapter 5

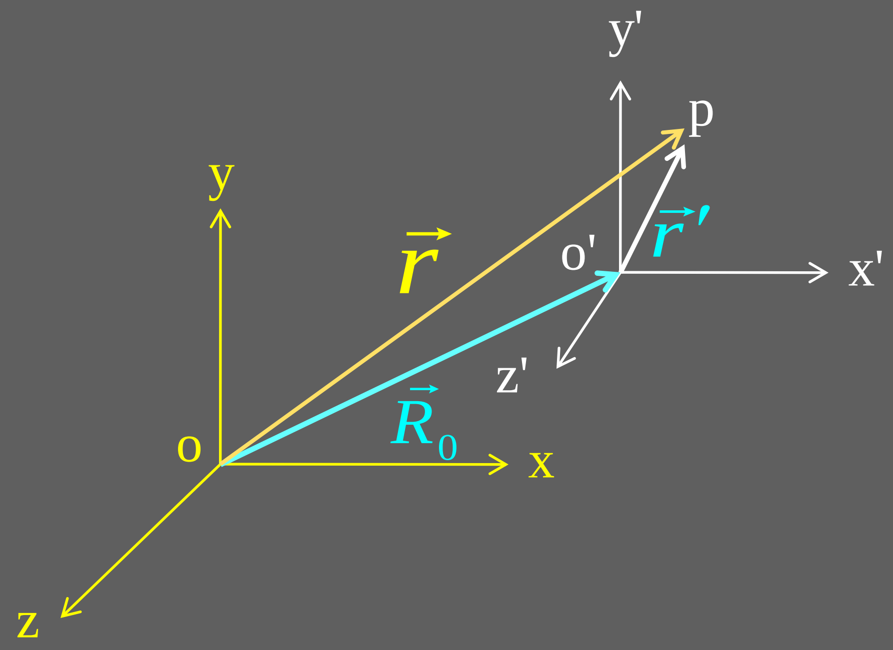
Noninertial Reference Systems

Accelerated Coordinate Systems and Inertial Forces

$$\vec{r} = \vec{R}_0 + \vec{r}'$$

$$\vec{v} = \vec{V}_0 + \vec{v}'$$

$$\vec{a} = \vec{A}_0 + \vec{a}'$$



$$\vec{A}_0 = 0 \rightarrow \vec{a} = \vec{a}'$$

$$\vec{F} = m\vec{a} \qquad \rightarrow \vec{F} = m\vec{a}'$$

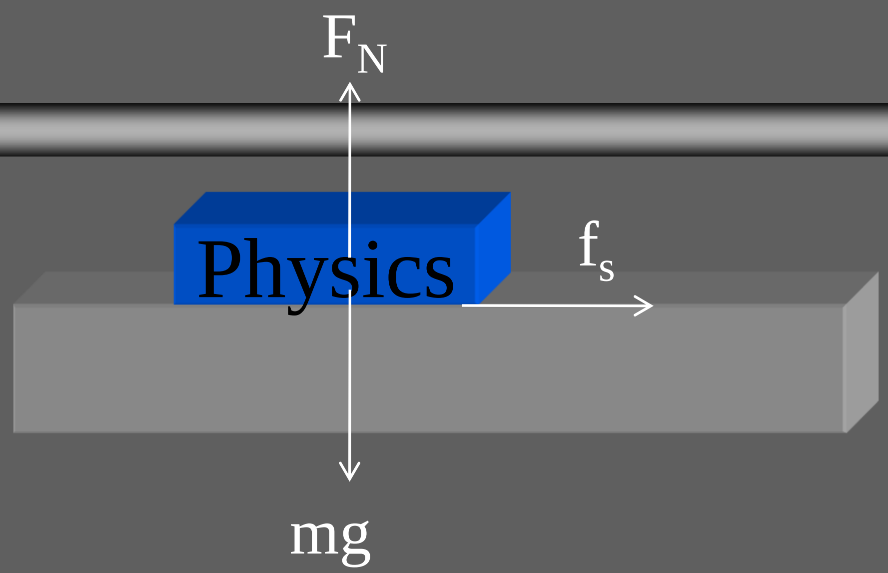
$$\vec{F} = m\vec{A}_0 + m\vec{a}'$$

$$\vec{F} - m\vec{A}_0 = m\vec{a}'$$

$$\vec{F}' = m\vec{a}'$$

$$\vec{F}' = \vec{F} + (-m\vec{A}_0)$$

Example



$$F_N - mg = 0$$

$$\mu_s mg = ma$$

$$\mu_s g = a$$

$$if \quad a > \mu_s g \rightarrow slip$$

$$\left| -m\vec{A}_0 \right| > \mu_s mg$$

$$\vec{F} - m\vec{A}_0 = m\vec{a}'$$

$$\mu_s mg = m\vec{A}_0$$

$$A_0 > \mu_s g$$

Rotating Coordinate Systems
Angular Velocity as a Vector Quantity

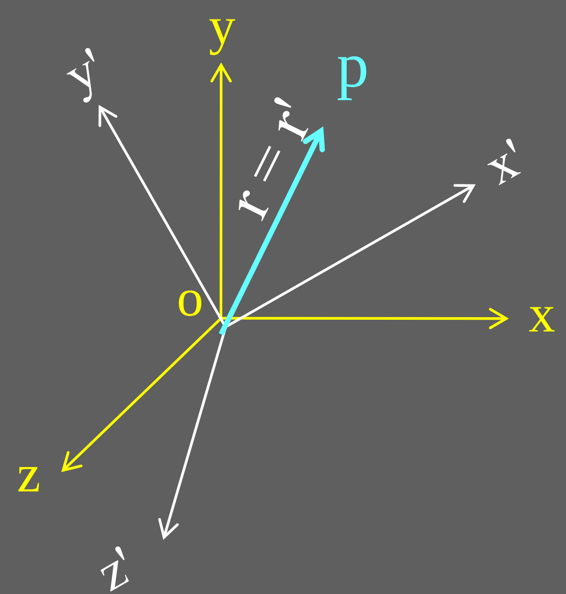
$$\vec{r} = \vec{r}'$$

$$\hat{i}x + \hat{j}y + \hat{k}z = \hat{i}'x' + \hat{j}'y' + \hat{k}'z'$$

$$\hat{i}\frac{dx}{dt} + \hat{j}\frac{dy}{dt} + \hat{k}\frac{dz}{dt}$$

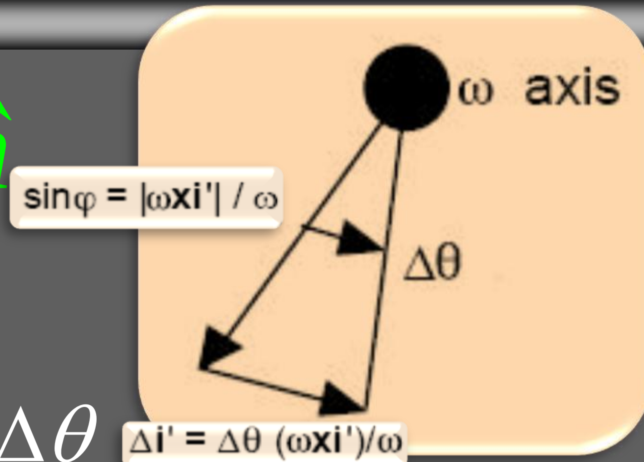
$$= \hat{i}'\frac{dx'}{dt} + \hat{j}'\frac{dy'}{dt} + \hat{k}'\frac{dz'}{dt}$$

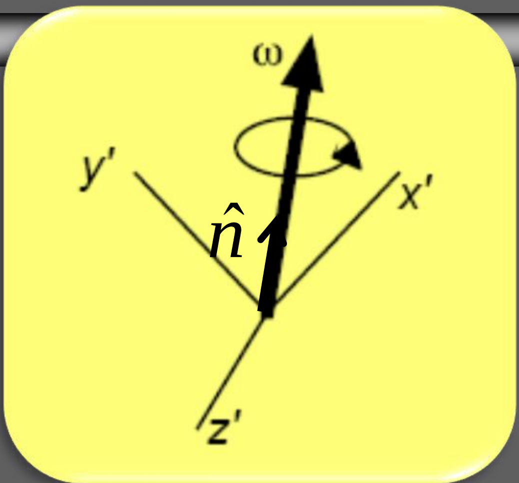
$$+ x'\frac{d\hat{i}'}{dt} + y'\frac{d\hat{j}'}{dt} + z'\frac{d\hat{k}'}{dt}$$



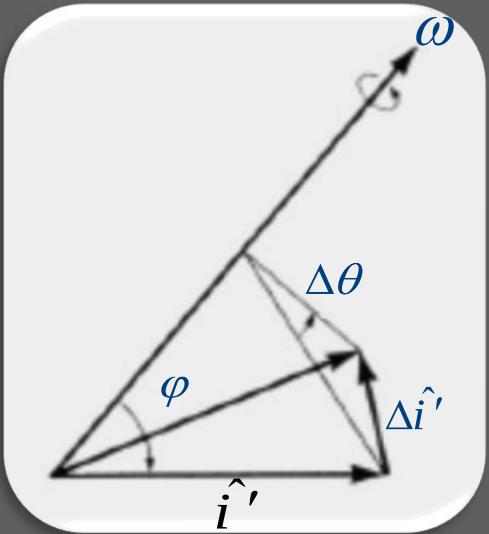
$$\vec{v} = \vec{v}' + x'\frac{d\hat{i}'}{dt} + y'\frac{d\hat{j}'}{dt} + z'\frac{d\hat{k}'}{dt}$$

$$\vec{\omega} = \omega\hat{n}$$

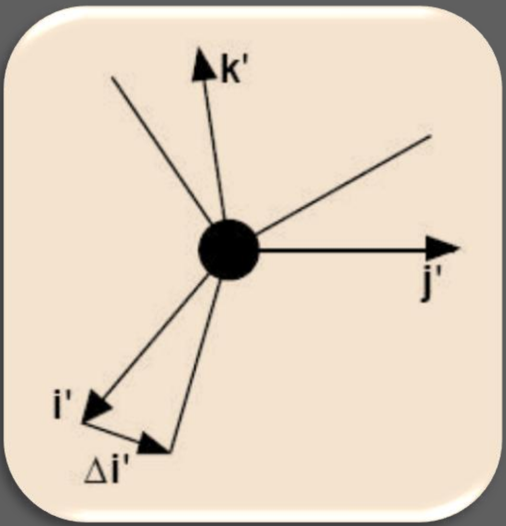




$$\left|\Delta\hat{i}'\right| \simeq (\sin\varphi)\Delta\theta$$



$$\Delta\hat{i}' \simeq \frac{\vec{\omega}\times\hat{i}}{\omega}\Delta\theta$$



$$\frac{d\hat{i}'}{dt} = \vec{\omega}\times\hat{i}'$$

$$\left|\frac{d\hat{i}'}{dt}\right| = \lim_{\Delta t \to 0}\left|\frac{\Delta\hat{i}'}{\Delta t}\right| = \sin\varphi\frac{d\theta}{dt} = (\sin\varphi)\omega$$

$$\frac{d\hat{i}'}{dt} = \vec{\omega}\times\hat{i}'$$

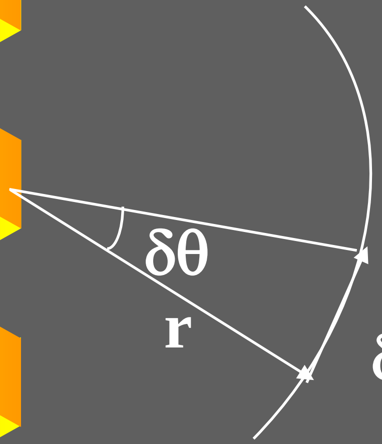
$$\frac{d\hat{j}'}{dt} = \vec{\omega}\times\hat{j}'$$

$$\frac{d\hat{k}'}{dt} = \vec{\omega}\times\hat{k}'$$

To interpret $\frac{d\hat{i}'}{dt}, \frac{d\hat{j}'}{dt}, \frac{d\hat{k}'}{dt}$

think of each unit vector as a position vector.

linear velocity = angular velocity x position vector



$$\vec{V} = \vec{\omega} \times \vec{r}$$

Because $\vec{V} = \frac{d\vec{r}}{dt}$ $\quad \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$

Thus $\frac{d\hat{i}'}{dt} = \vec{\omega} \times \hat{i}', \quad \frac{d\hat{j}'}{dt} = \vec{\omega} \times \hat{j}', \quad \frac{d\hat{k}'}{dt} = \vec{\omega} \times \hat{k}'$

$$x'\frac{d\hat{i}'}{dt}+y'\frac{d\hat{j}'}{dt}+z'\frac{d\hat{k}'}{dt}=x'(\vec{\omega}\times\hat{i}')+y'(\vec{\omega}\times\hat{j}')+z'(\vec{\omega}\times\hat{k}')$$

$$=\vec{\omega}\times(\hat{i}'x'+\hat{j}'y'+\hat{k}'z')=\vec{\omega}\times\vec{r}'$$

$$\vec{v}=\vec{v}'+\vec{\omega}\times\vec{r}'$$

$$\left(\frac{d\vec{r}}{dt}\right)_{inertial}=\left(\frac{d\vec{r}'}{dt}\right)_{rot}+\vec{\omega}\times\vec{r}'=\left[\left(\frac{d}{dt}\right)_{rot}+\vec{\omega}\times\right]\vec{r}'$$

$$\left(\frac{d\ldots}{dt}\right)_{inertial}=\left(\frac{d\ldots}{dt}\right)_{rot}+\vec{\omega}\times\cdots$$

$$\left(\frac{d\vec{v}}{dt}\right)_{inertial} = \left(\frac{d\vec{v}}{dt}\right)_{rot} + \vec{\omega}\times\vec{v}$$

$$\vec{v} = \vec{v}' + \vec{\omega}\times\vec{r}'$$

$$\left(\frac{d\vec{v}}{dt}\right)_{inertial} = \left(\frac{d}{dt}\right)_{rot}(\vec{v}' + \vec{\omega}\times\vec{r}') + \vec{\omega}\times(\vec{v}' + \vec{\omega}\times\vec{r}')$$

$$= \left(\frac{d\vec{v}'}{dt}\right)_{rot} + \left[\frac{d(\vec{\omega}\times\vec{r}')}{dt}\right]_{rot} + \vec{\omega}\times\vec{v}' + \vec{\omega}\times(\vec{\omega}\times\vec{r}')$$

$$= \left(\frac{d\vec{v}'}{dt}\right)_{rot} + \left(\frac{d\vec{\omega}}{dt}\right)_{rot}\times\vec{r}' + \vec{\omega}\times\left(\frac{d\vec{r}'}{dt}\right)_{rot} + \vec{\omega}\times\vec{v}' + \vec{\omega}\times(\vec{\omega}\times\vec{r}')$$

$$\left(\frac{d\vec{\omega}}{dt}\right)_{inertial} = \left(\frac{d\vec{\omega}}{dt}\right)_{rot} + \vec{\omega}\times\vec{\omega}$$

$$\vec{\omega} \times \vec{\omega} = 0$$

$$\left(\frac{d\,\vec{\omega}}{dt}\right)_{inertial} = \left(\frac{d\,\vec{\omega}}{dt}\right)_{rot} = \dot{\omega}$$

$$\vec{v}\,' = \left(\frac{d\vec{r}\,'}{dt}\right)_{rot} \qquad \vec{a}\,' = \left(\frac{d\vec{v}\,'}{dt}\right)_{rot}$$

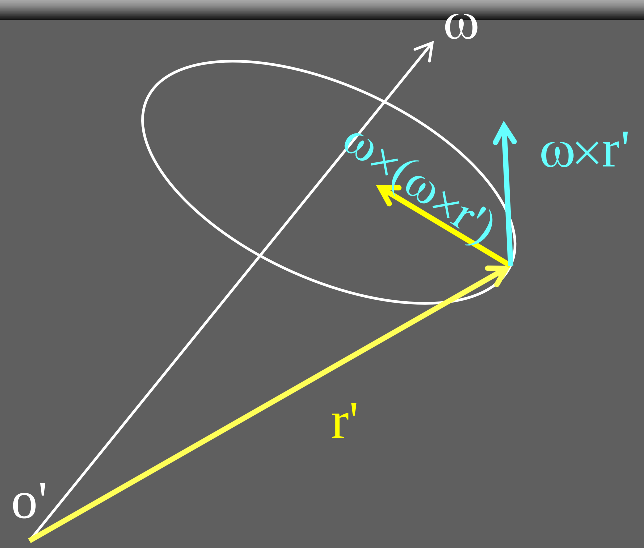
$$\vec{a} = \vec{a}\,' + \vec{\dot{\omega}} \times \vec{r}\,' + 2\vec{\omega} \times \vec{v}\,' + \vec{\omega} \times (\vec{\omega} \times \vec{r}\,')$$

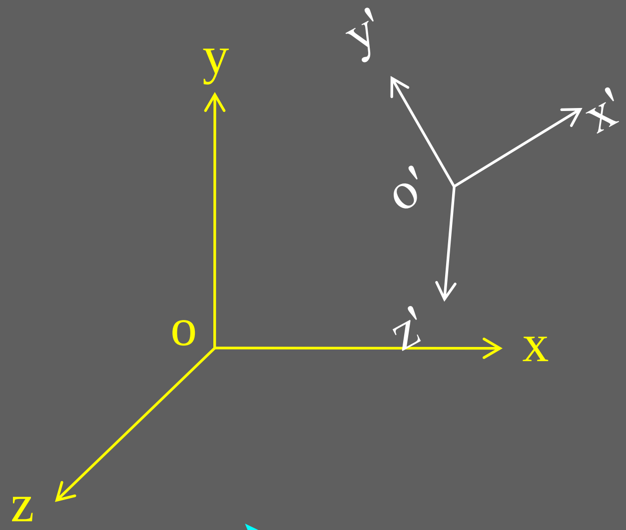
transverse acceleration

coriolis acceleration

centripetal acceleration

In the general case in which the primed system in undergoing both translation and rotation

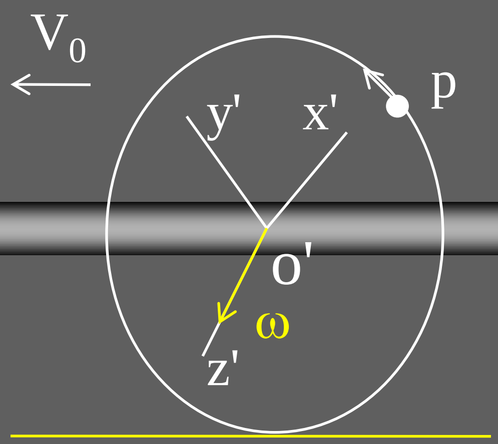




$$\vec{v} = \vec{v}' + \vec{\omega} \times \vec{r}' + \vec{V}_0$$

$$\vec{a} = \vec{a}' + \dot{\vec{\omega}} \times \vec{r}' + 2\vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}') + \vec{A}_0$$

Example1:



$$\vec{r}' = b\hat{i}' \qquad \vec{a}' = \ddot{\vec{r}} = 0$$

$$\vec{v}' = \dot{\vec{r}} = 0$$

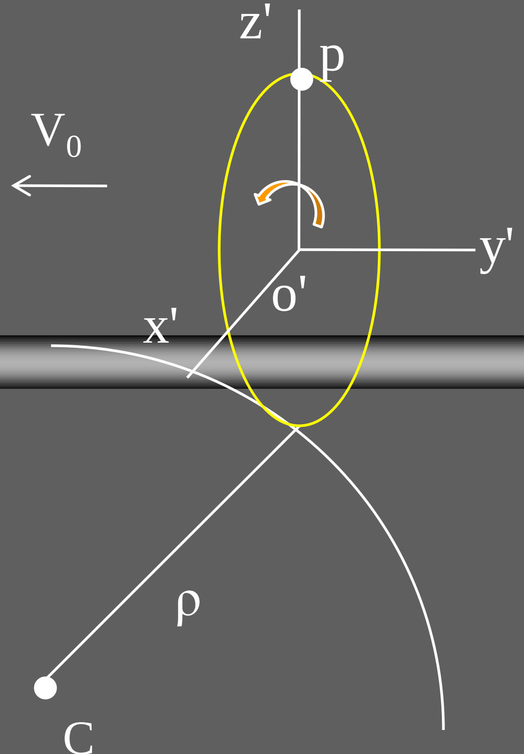
$$\vec{\omega} = \omega\hat{k}' = \frac{V_0}{b}\hat{k}'$$

$$\vec{a} = \vec{a}' + \dot{\vec{\omega}} \times \vec{r}' + 2\vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}') + \vec{A}_0$$

$$\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{r}') = \frac{V_0}{b}\hat{k}' \times \left(\frac{V_0}{b}\hat{k}' \times b\hat{i}'\right)$$

$$= \frac{V_0^2}{b}\hat{k}' \times (\hat{k}' \times \hat{i}') = \frac{V_0^2}{b}\hat{k}' \times \hat{j}' = \frac{V_0^2}{b}(-\hat{i}')$$

Example 2:



$$\vec{r}' = b\hat{k}' \qquad \vec{\omega} = \frac{V_0}{\rho}\hat{k}'$$

$$\vec{A}_0 = \frac{V_0^2}{\rho}\hat{i}' \qquad \ddot{\vec{r}}' = -\frac{V_0^2}{b}\hat{k}'$$

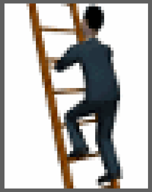
$$\vec{v}' = -V_0\hat{j}'$$

$$\vec{a} = \vec{a}' + \dot{\vec{\omega}}\times\vec{r}' + 2\vec{\omega}\times\vec{v}' + \vec{\omega}\times(\vec{\omega}\times\vec{r}') + \vec{A}_0$$

$$2\vec{\omega}\times\vec{v}' = 2\left(\frac{V_0}{\rho}\hat{k}'\right)\times(-V_0\hat{j}') = 2\frac{V_0^2}{\rho}\hat{i}'$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{r}') = \frac{V_0^2}{\rho^2}\hat{k}'\times(\hat{k}'\times b\hat{k}') = 0$$

$$\vec{a} = 3\frac{V_0^2}{\rho}\hat{i}' - \frac{V_0^2}{b}\hat{k}'$$



Dynamics of a Particle in a Rotating Coordinate System

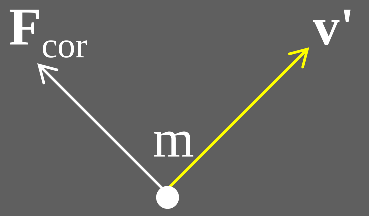
$$\vec{F} = m\vec{a}$$

$$\vec{a} = \vec{a}' + \dot{\vec{\omega}} \times \vec{r}' + 2\vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}') + \vec{A}_0$$

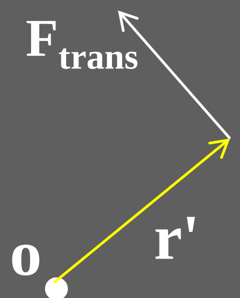
$$\vec{F} - m\vec{A}_0 - 2m\vec{\omega} \times \vec{v}' - m\dot{\vec{\omega}} \times \vec{r}' - m\vec{\omega} \times (\vec{\omega} \times \vec{r}') = m\vec{a}'$$

$$\vec{F}' = m\vec{a}' \qquad \vec{F}' = \vec{F} + \vec{F}_{cor} + \vec{F}_{trans} + \vec{F}_{cent} - m\vec{A}_0$$

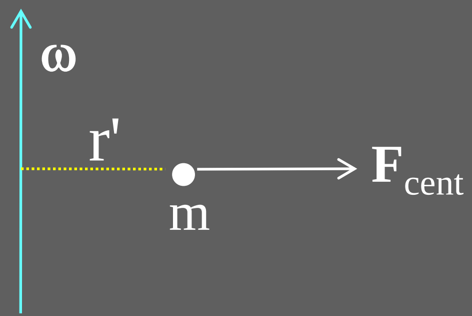
$$F_{cor} = -2m\vec{\omega} \times \vec{v}'$$



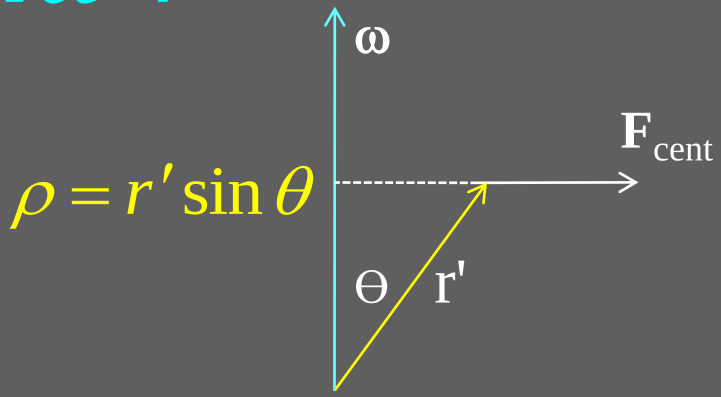
$$\vec{F}_{trans} = -m\dot{\vec{\omega}} \times \vec{r}'$$



$$\vec{F}_{cent} = -m\vec{\omega}\times(\vec{\omega}\times\vec{r}')$$



$$\left|m\vec{\omega}\times(\vec{\omega}\times\vec{r}')\right| = m\omega^2 r$$



$$\left|m\vec{\omega}\times(\vec{\omega}\times\vec{r}')\right| = m\,|\vec{\omega}|\,|(\vec{\omega}\times\vec{r}')|\sin 90$$

$$= m\,|\vec{\omega}|\,|\vec{\omega}|\,|\vec{r}'|\sin\theta = m\omega^2\rho$$