ANALYTICAL MECHANICS 1

Lecture 14

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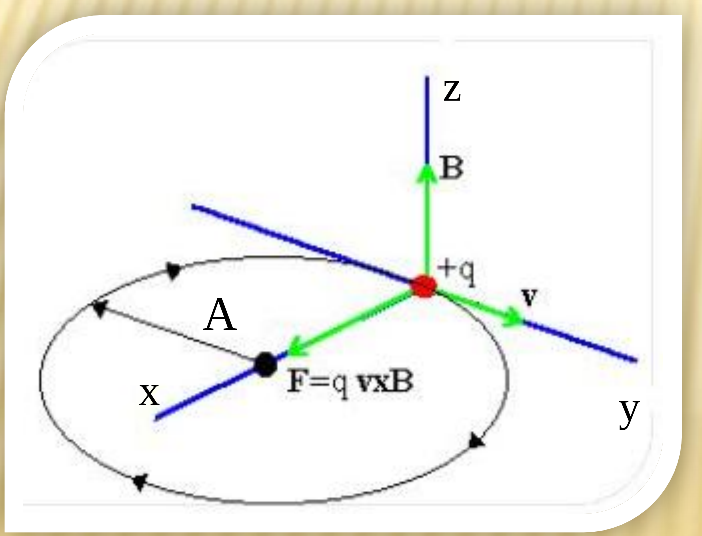
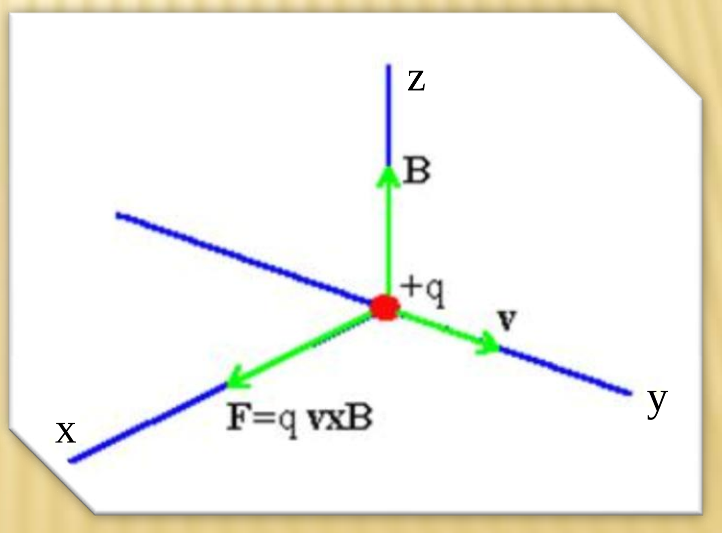
When the magnetic field is perpendicular to the velocity, the motion will clearly be circular since the force will always be oriented 'inward' perpendicular to the velocity - it is a centripetal force which leads to a centripetal acceleration. The trajectory, for a positive charge, is shown below:

$$\vec{F} = q\vec{v} \times \vec{B} = qv\hat{j} \times B\hat{k} = qvB\hat{i}$$

$$\dot{z} = 0$$

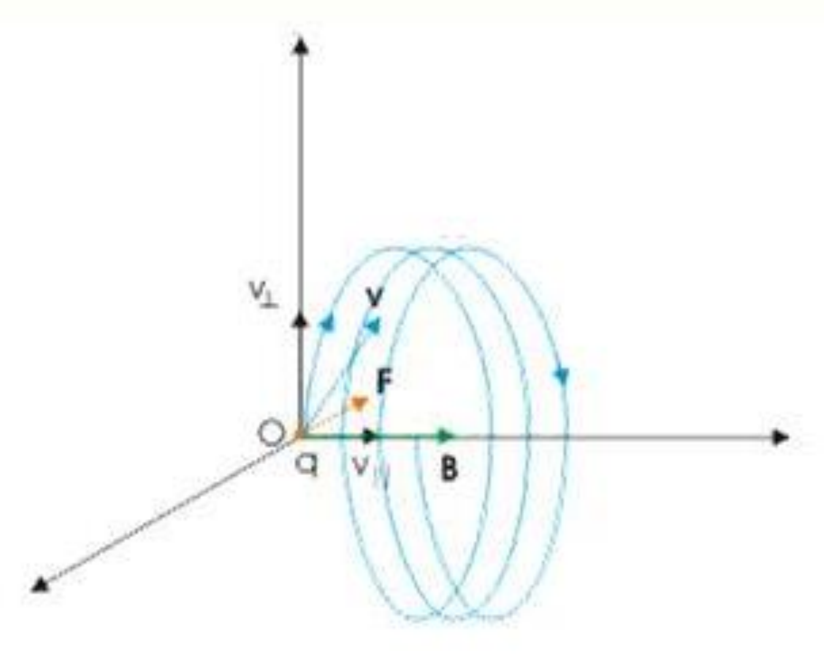




What happens if the velocity is not perpendicular to the field? In this case, we can resolve the velocity into two components, one parallel to the field and one perpendicular.

$$\vec{F} = q\vec{v} \times \vec{B} = q(\vec{v}_{\parallel} + \vec{v}_{\perp}) \times \vec{B} = q\vec{v}_{\perp} \times \vec{B}$$

This perpendicular component of velocity varies in the same way as the previous example of circular motion. There will be no force along the magnetic field, so the motion in that direction is simply a constant velocity given by the parallel component of velocity to the field - this component is unaffected by the magnetic field. In this case, the motion will be helical:



$$\dot{z} = v_{\parallel}$$

Constrained Motion of a Particle

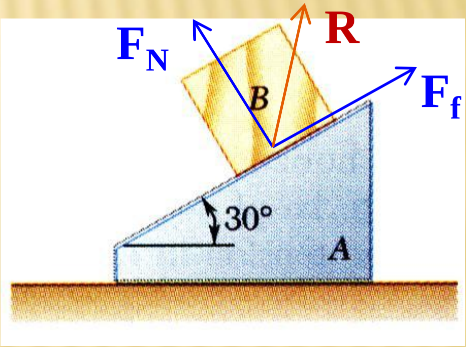
When a moving particle in restricted geometrically in the sense that is must stay on a certain definite surface or curve, the motion is said to be constrained

The Energy Equation for Smooth Constrains

$$m\frac{d\vec{v}}{dt} = \vec{F} + \vec{R}$$

$$m\frac{d\vec{v}}{dt} . \vec{v} = \vec{F} . \vec{v} + \vec{R} . \vec{v}$$

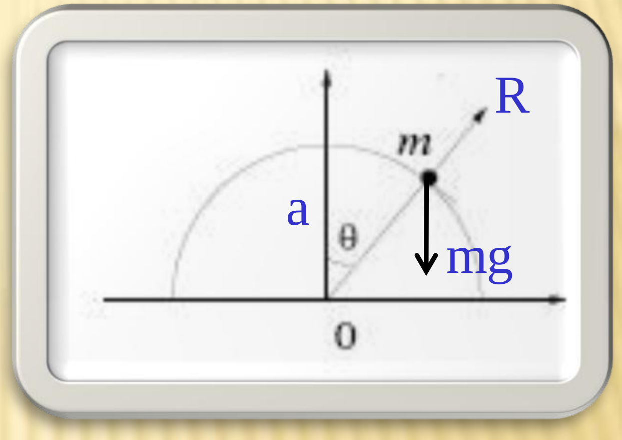
$$\frac{d}{dt}\left(\frac{1}{2} m \vec{v} . \vec{v}\right) = \vec{F} . \vec{v}$$



$$\frac{1}{2} m v^2 + V(x, y, z) = const. = E$$

Example 1: *A particle is placed on top of a smooth sphere of radius a. If the particle is slightly disturbed, at what point will it leave the sphere.*

$$m\frac{d\vec{v}}{dt} = m\vec{g} + \vec{R}$$



$$\frac{1}{2}mv^2 + mgz = E$$

$$v = 0 \;\; for \;\; z = a \rightarrow E = mga$$

$$v^2 = 2g(a - z)$$

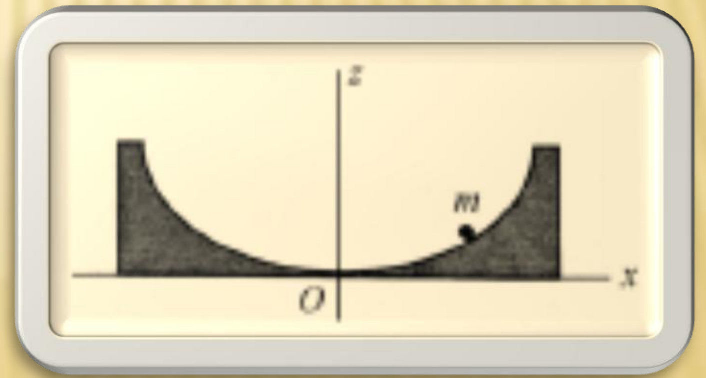
$$-\frac{mv^2}{a} = -mg\cos\theta + R = -mg\frac{z}{a} + R$$

$$R = mg\frac{z}{a} - \frac{mv^2}{a} = mg\frac{z}{a} - \frac{m}{a}2g(a-z)$$

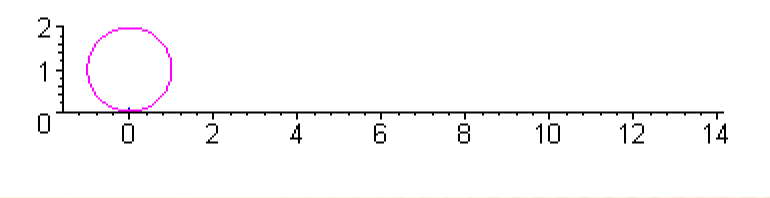
$$R = \frac{mg}{a}(3z - 2a) \qquad when\ z = \frac{3}{2}a \rightarrow R = 0$$

Example 2: *Constrained motion can a cycloid. Consider a particle sliding under gravity in a smooth cycloidal trough, Figure, represented by the parametric equations.*

$$\begin{cases} x = A(2\varphi + \sin 2\varphi) \\ z = A(1 - \cos 2\varphi) \end{cases}$$



A cycloid is the locus of a point on the circumference of a circle rotating along a fixed line . . . .



Now the energy equation for the motion, assuming no-y-motion, is:

$$E = \frac{1}{2}mv^2 + V(z) = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) + mgz$$

$$\begin{cases} \dot{x} = 2A\dot{\varphi}(1+\cos 2\varphi) \\ \dot{z} = 2A\dot{\varphi}\sin 2\varphi \end{cases}$$

$$E = 4mA^2\dot{\varphi}^2(1+\cos 2\varphi) + mgA(1-\cos 2\varphi)$$

$$we\ \ have:\ 1+\cos 2\varphi = 2\cos^2\varphi)$$

$$1-\cos 2\varphi = 2\sin^2\varphi)$$

$$E = 8mA^2\dot{\varphi}^2\cos^2\varphi + 2mgA\sin^2\varphi$$

$$s = 4A\sin\varphi \qquad \dot{s} = 4A\dot{\varphi}\cos\varphi$$

$$E = 8mA^2\dot{\varphi}^2\frac{\dot{s}^2}{16A^2\dot{\varphi}^2} + 2mgA\left(\frac{s^2}{16A^2}\right)$$

$$E = \frac{m}{2}\dot{s}^2 + \frac{1}{2}\left(\frac{mg}{4A}\right)s^2$$

$$0 = m\dot{s}\ddot{s} + \frac{mg}{4A}\dot{s}s \qquad \ddot{s} + \frac{g}{4A}s = 0 \qquad \ddot{x} + \frac{k}{m}x = 0$$

problem No 18

$$\ddot{s} + \frac{g}{4A} s = 0$$

$$\omega = \sqrt{\frac{g}{4A}}$$

$$\frac{2\pi}{T} = \sqrt{\frac{g}{4A}}$$

$$T = 2\pi\sqrt{\frac{4A}{g}}$$

$$T = 4\pi\sqrt{\frac{A}{g}}$$

4) $V(x,y,z)=\alpha x+\beta y^2+\gamma z^3$

a) The total energy E equals to the initial kinetic energy

$$E=\frac{1}{2}mv_0^2$$

When the particle is at the point (1, 1, 1) its potential energy is

$$V=\alpha+\beta+\gamma$$

$$E=\frac{1}{2}mv_0^2=\alpha+\beta+\gamma+\frac{1}{2}mv^2$$

$$v^2=v_0^2-\frac{2}{m}(\alpha+\beta+\gamma)$$

b) $v_0^2=\frac{2}{m}(\alpha+\beta+\gamma)$

c) $$V(x,y,z) = \alpha x + \beta y^2 + \gamma z^3$$

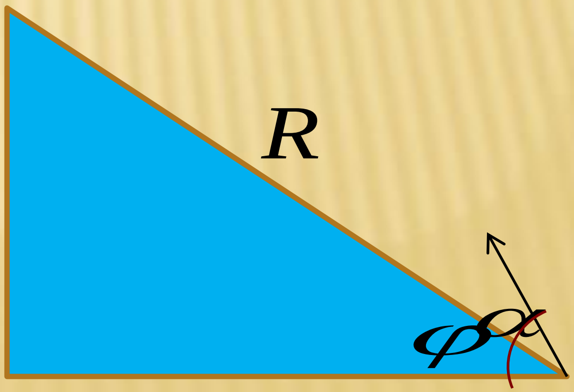
$$m\ddot{x} = -\frac{\partial V}{\partial x} = -\alpha$$

$$m\ddot{y} = -\frac{\partial V}{\partial y} = -2\beta y \qquad m\ddot{z} = -\frac{\partial V}{\partial z} = -3\gamma z^2$$

Problem 4.8: Let R is the distance measured along the slope between the initial and final points of the projectile trajectory. It is what we have to find. Let x and y are horizontal and vertical coordinate of the final point. Then

$$x = R\cos\varphi \qquad y = R\sin\varphi$$

$$x = v_x t \qquad y = v_y t - \frac{1}{2} g t^2$$



$$R\cos\varphi = v_x t \qquad R\sin\varphi = v_y t - \frac{1}{2}gt^2$$

Eliminating from here t one gets

$$R = \frac{2v_0^2\cos\alpha(\sin\alpha\cos\varphi - \cos\alpha\sin\varphi)}{g\cos^2\varphi}$$

$$\sin(\alpha-\varphi) = \sin\alpha\cos\varphi - \cos\alpha\sin\varphi$$

$$R = \frac{2v_0^2\cos\alpha\sin(\alpha-\varphi)}{g\cos^2\varphi}$$

R as a function of the angle α of gun elevation has maximum when $dR/d\alpha = 0$ which gives the condition

$$-\sin\alpha\sin(\alpha-\varphi) + \cos\alpha\cos(\alpha-\varphi) = 0$$

$$\sin(\theta - \varphi) = \sin\theta\cos\varphi - \cos\theta\sin\varphi$$

$$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$$

$$\cos(2\alpha - \varphi) = 0$$

$$2\alpha - \varphi = \pi / 2 \qquad \alpha = \varphi / 2 + \pi / 4$$

Substituting $\alpha = \phi/2 + \pi/4$ into the above formula for R and using trigonometric identities from the appendix B one can get the desirable result

$$R_{\max} = \frac{v_0^{\,2}}{g(1+\sin\varphi)}$$

11) $\ddot{x}(t) = -\omega^2 x(t)$ $\ddot{y}(t) = -\omega^2 y(t)$

The general solutions of these equations can be written for example as

$$x(t) = a_1 \cos \omega t + a_2 \sin \omega t$$

$$y(t) = b_1 \cos \omega t + b_2 \sin \omega t$$

where the constants a_i and b_i must be found from initial conditions. The initial conditions involves velocities so we need to know the derivatives

$$\dot{x}(t) = -a_1 \omega \sin \omega t + a_2 \omega \cos \omega t$$

$$\dot{y}(t) = -b_1 \omega \sin \omega t + b_2 \omega \cos \omega t$$

$$t = 0 \qquad x(0) = A \qquad \dot{x}(0) = 0$$

$$a_1 = A \qquad a_2\omega = 0$$

$$x(t) = A\cos\omega t$$

$$y(0) = 4A \qquad \dot{y}(0) = 3\omega A$$

$$b_1 = 4A \qquad b_2\omega = 3\omega A$$

$$y(t) = 4A\cos\omega t + 3A\sin\omega t$$

$$x_{\max} = A$$

The matter with y_{max} is a little more complicated. Let t_m is a moment when $y(t)$ take a maximum, i.e.

$$y_{\max} = y(t_m)$$

t_m can be found from the condition $\dot{y}(t_m) = 0$ which gives

$$\tan \omega t_m = 3/4$$

$$y_{\max} = y(t_m) = 4A\cos\omega t_m + 3A\sin\omega t_m$$

$$\sin z = \frac{\tan z}{\sqrt{1+\tan^2 z}} \quad \cos z = \frac{1}{\sqrt{1+\tan^2 z}} \quad y_{\max} = 5A$$

Therefore the motion occurs in the rectangle

$$-A < x < A$$

$$-5A < y < 5A$$

Instead of writing the general solutions of equations of motion in the form

$$x(t) = a_1 \cos\omega t + a_2 \sin\omega t$$

$$y(t) = b_1 \cos \omega t + b_2 \sin \omega t$$

$$x(t) = a\cos(\omega t + \alpha) \quad y(t) = b\cos(\omega t + \beta)$$

$$t = 0 \;\; a = A, \quad \alpha = 0 \qquad t = 0 \rightarrow y = 4A = 5A\cos\beta$$

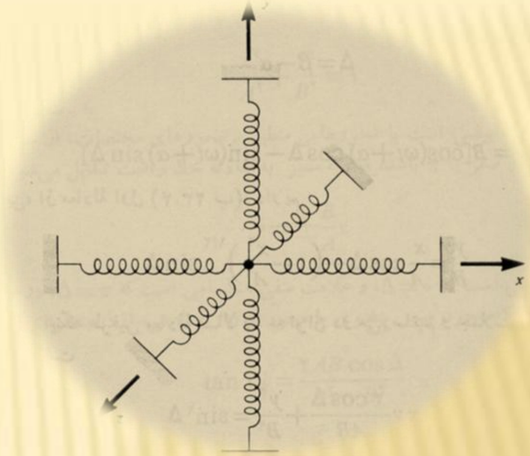
$$\beta = \cos^{-1}(4/5) = \sin^{-1}(-3/5)$$

$$tg\,2\psi = \frac{2ab\cos\Delta}{a^2 - b^2} \qquad Here \;\; \Delta = \beta - \alpha = \beta$$

$$tg\,2\psi = \frac{2A.5A.4/5}{A^2 - (5A)^2} = -1/3$$

$$\psi = \frac{1}{2}\tan^{-1}(-1/3) = -9.2^\circ$$

12) $1:4:9 \quad V = \frac{k}{2}(x^2 + 4y^2 + 9z^2)$



$$m\ddot{x} = -\frac{\partial V}{\partial x} = -kx \rightarrow \omega_x = \sqrt{\frac{k}{m}} = \omega$$

$$m\ddot{y} = -\frac{\partial V}{\partial y} = -4ky \rightarrow \omega_y = \sqrt{\frac{4k}{m}} = 2\omega$$

$$m\ddot{z} = -\frac{\partial V}{\partial z} = -9kz \rightarrow \omega_z = \sqrt{\frac{9k}{m}} = 3\omega$$

$$\ddot{x} = -\omega_x{}^2 x \qquad \ddot{y} = -\omega_y{}^2 y \qquad \ddot{z} = -\omega_z{}^2 z$$

$$t=0\rightarrow x(0)=y(0)=z(0)=0$$

$$v_0^{\,2}=\dot{x}^2(0)+\dot{y}^2(0)+\dot{z}^2(0)$$

$$\dot{x}(0)=\dot{y}(0)=\dot{z}(0)=\frac{v_0}{\sqrt{3}}$$

$$x(t)=A\cos(\omega t+\alpha)\quad x(t)=A\cos(\omega t-\frac{\pi}{2})$$

$$\dot{x}(t)=A\omega\sin(\omega t-\frac{\pi}{2})\quad\rightarrow\frac{v_0}{\sqrt{3}}=A\omega$$

$$\omega_x^{\,2}=\frac{k}{m}\rightarrow\omega_x=\pi\qquad x(t)=\frac{v_0}{\pi\sqrt{3}}\sin\pi t$$

$$y(t)=\frac{v_0}{2\pi\sqrt{3}}\sin 2\pi t\qquad z(t)=\frac{v_0}{3\pi\sqrt{3}}\sin 3\pi t$$

$$\omega_x = \pi, \quad \omega_y = 2\pi, \quad \omega_z = 3\pi$$

$$\frac{\omega_x}{n_x} = \frac{\omega_x}{n_y} = \frac{\omega_x}{n_z}$$

$$n_x = 1 \quad n_y = 2 \quad n_z = 3$$

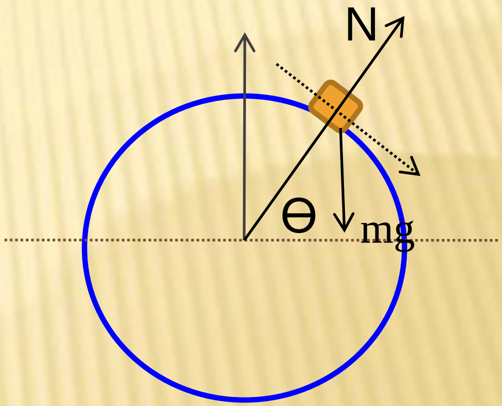
$$t = \frac{2\pi n_x}{\omega_x} = \frac{2\pi n_y}{\omega_y} = \frac{2\pi n_z}{\omega_z}$$

$$t = \frac{2\pi}{\pi} = 2$$

16)

$$N - mg\cos(90-\theta) = -m\frac{v^2}{b}$$

$$-N + mg\sin\theta = m\frac{v^2}{b}$$



but with N = 0, one has the velocity at which N = 0

$$v^2 = bg\sin\theta$$

(When this velocity is achieved N will become zero.)

$$mg\frac{b}{2} = \frac{1}{2}m(bg\sin\theta) + mgb\sin\theta$$

$$mg\frac{b}{2} = \frac{3}{2}mbg\sin\theta \rightarrow \sin\theta = \frac{1}{3} \quad \rightarrow h = \frac{b}{3}$$

Example

$$\vec{F}(v) - \vec{v}(c_1 + c_2|v|)$$

$$\gamma = \frac{c_1 + c_2 v_0}{m} \quad m = 0.046kg, \qquad D = 0.042m$$

$$\gamma = \frac{c_2 v_0}{m} = \frac{0.22D^2 v_0}{m} = 0.0084 v_0$$

$$v_0 = 20m/s \rightarrow \gamma = 0.17s^{-1}$$

$$x_h = \frac{v_0^2 \sin 2\alpha}{g} - \frac{4v_0^3 \sin 2\alpha \sin\alpha}{3g^2}\gamma + \ldots$$

$$x_h = \frac{(20)^2 \sin 60^\circ}{9.8} - \frac{4(20)^3 \sin 60^\circ \sin 30^\circ \times 0.17}{3(9.8)^2}$$

$$= 35.3m - 8.2m = 27.1m$$