ANALYTICAL MECHANICS 1

Lecture 13

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The Two Dimension Isotropic Oscillator

$$m\ddot{x} = -kx \qquad m\ddot{y} = -ky$$

$$\omega = (k/m)^{1/2}$$

$$x = A\cos(\omega t + \alpha) \qquad y = B\cos(\omega t + \beta)$$

$$\Delta = \beta - \alpha$$

$$\frac{x^2}{A^2} - \frac{2\cos\Delta}{AB}xy + \frac{y^2}{B^2} = \sin^2\Delta$$

$$ax^2 + bxy + cy^2 + dx + ey = f$$

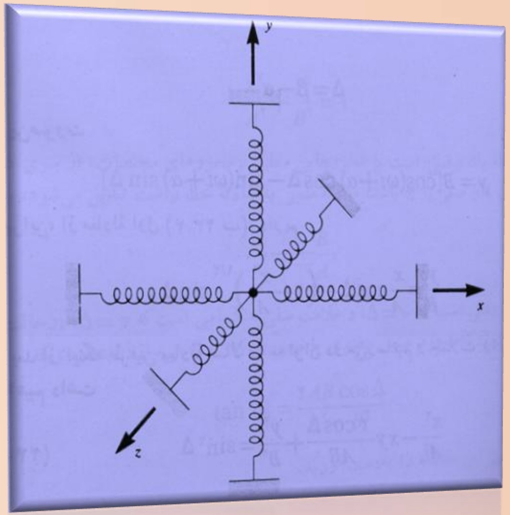
$$\text{if } \Delta = 0, \quad \Delta = \pi, \quad \Delta = \pi/2$$

The Three Dimension Isotropic Harmonic Oscillator

$$m\ddot{x} = -kx$$

$$m\ddot{y} = -ky$$

$$m\ddot{z} = -kz$$



$$x = A_1 \sin \omega t + B_1 \cos \omega t$$

$$y = A_2 \sin \omega t + B_2 \cos \omega t$$

$$z = A_3 \sin \omega t + B_3 \cos \omega t$$

$$\vec{r} = \vec{A}\sin\omega t + \vec{B}\cos\omega t$$

$$t = 0 \rightarrow \begin{cases} x = x_0 \\ y = y_0 \\ z = z_0 \end{cases} \rightarrow \begin{cases} \dot{x} = \dot{x}_0 \\ \dot{y} = \dot{y}_0 \\ \dot{z} = \dot{z}_0 \end{cases}$$

Non-isotropic Oscillator

$$m\ddot{x} = -k_1 x \qquad \omega_1 = (k_1/m)^{1/2}$$

$$m\ddot{y} = -k_2 y \qquad \omega_2 = (k_2/m)^{1/2}$$

$$m\ddot{z} = -k_3 z \qquad \omega_3 = (k_3/m)^{1/2}$$

$$x = A\cos(\omega_1 t + \alpha)$$

$$y = B\cos(\omega_2 t + \beta)$$

$$z = C\cos(\omega_3 t + \gamma)$$

$$\frac{\omega_1}{n_1} = \frac{\omega_2}{n_2} = \frac{\omega_3}{n_3}$$

$$T = \frac{2\pi n_1}{\omega_1} = \frac{2\pi n_2}{\omega_2} = \frac{2\pi n_3}{\omega_3}$$

Energy Consideration

$$V(x) = \frac{1}{2} k x^2$$

$$V(x, y, z) = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 y^2 + \frac{1}{2} k_3 z^2$$

$$F_x = -\frac{\partial V}{\partial x} = -k_1 x$$

$$k_1 = k_2 = k_3 = k$$

$$V(x, y, z) = \frac{1}{2} k (x^2 + y^2 + z^2) = \frac{1}{2} k r^2$$

$$\frac{1}{2}mv^2 + \frac{1}{2}kr^2 = E$$

Example: A particle of mass m moves in two dimension under the following potential energy function

$$V(r) = \frac{1}{2}k(x^2 + 4y^2)$$

Find the resulting motion.

Given the initial condition at

$$t = 0: \; x = a, \; y = 0, \; \dot{x} = 0, \; \dot{y} = v_0$$

$$\vec{F} = -\nabla V = -\hat{i}kx - \hat{j}\,4ky = m\ddot{\vec{r}}$$

The component differential equation of motion are then.

$$m\ddot{x} + kx = 0 \qquad m\ddot{y} + 4ky = 0$$

$$\omega_x = (k/m)^{1/2} = \omega \qquad \omega_y = (4k/m)^{1/2} = 2\omega$$

$$x = A_1 \cos\omega t + B_1 \sin\omega t$$

$$y = A_2 \cos 2\omega t + B_2 \sin 2\omega t$$

$$\dot{x} = -A_1\omega \sin\omega t + B_1\omega \cos\omega t$$

$$\dot{y} = -2A_2\omega \sin 2\omega t + 2B_2\omega \cos 2\omega t$$

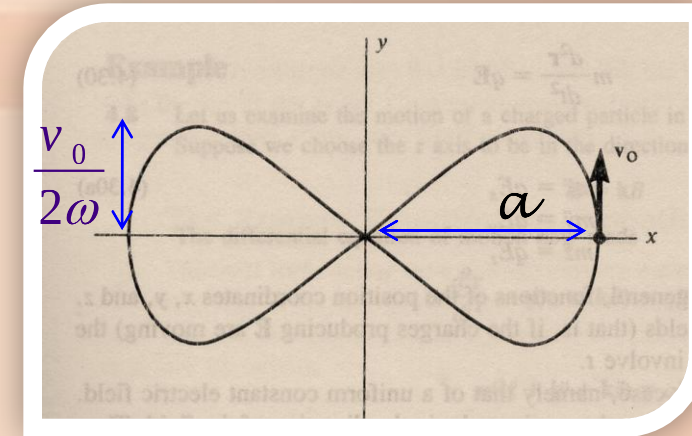
$$t = 0:\; x = a,\; y = 0,\; \dot{x} = 0,\; \dot{y} = v_0$$

$$a = A_1 \qquad 0 = A_2 \qquad 0 = B_1\omega \qquad v_0 = 2B_2\omega$$

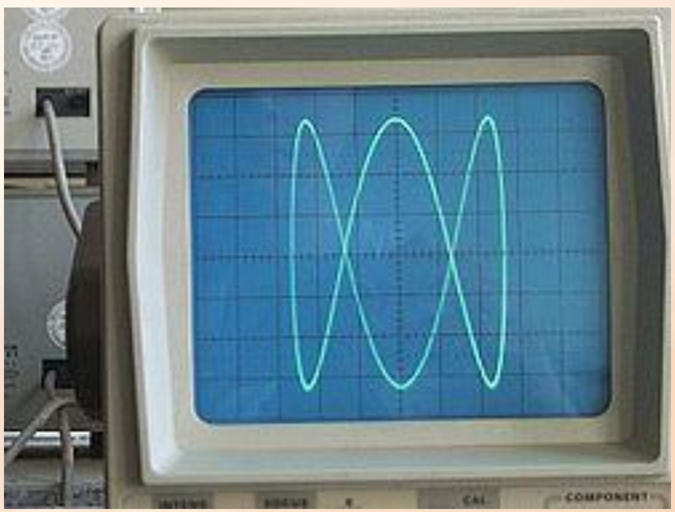
$$A_1 = a \qquad A_2 = B_1 = 0 \qquad B_2 = v_0 / 2\omega$$

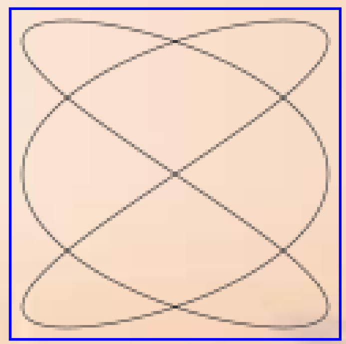
$$x = a\cos\omega t$$

$$y = \frac{v_0}{2\omega}\sin 2\omega t$$

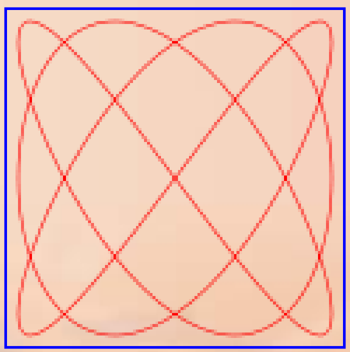


$$\omega_y = 2\omega_x$$

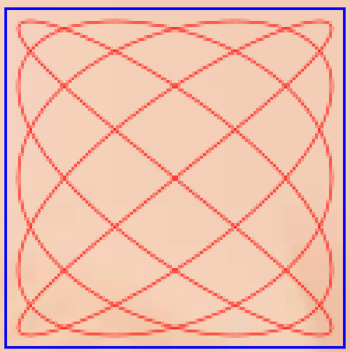




$2\omega_x = 3\omega_y$



$4\omega_x = 3\omega_y$



$4\omega_x = 5\omega_y$

Motion of Charged Particles in Electric and Magnetic Fields

Charges at rest → electric field $\vec{F} = q\vec{E}$ $m\dfrac{d^2\vec{r}}{dt^2} = q\vec{E}$

$$m\ddot{x} = qE_x \qquad m\ddot{y} = qE_y \qquad m\ddot{z} = qE_z$$

$$E_x = E_y = 0, \qquad E = E_z$$

$$\ddot{x} = 0 \qquad \ddot{y} = 0 \qquad \ddot{z} = \frac{qE}{m} = cons.$$

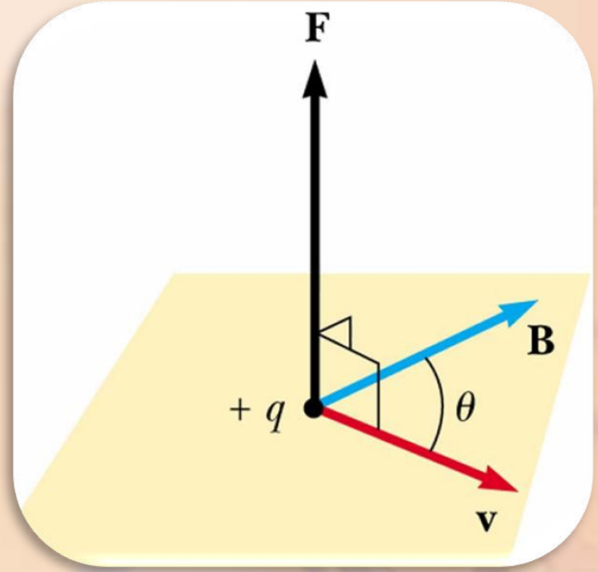
$$\dot{z} = \frac{qE}{m}t + \dot{z}_0 \qquad z = \frac{qE}{2m}t^2 + \dot{z}_0 t + z_0$$

$$\nabla \times \vec{E} = 0 \qquad \vec{E} = -\nabla \varphi$$

$$\frac{1}{2} m v^2 + q^{\varphi} = const.$$

Moving charges → magnetic field $\vec{F} = q(\vec{v} \times \vec{B})$

$$m \frac{d^2 \vec{r}}{dt^2} = q(\vec{v} \times \vec{B})$$



Example: Let us examine the motion of a charge particle in a uniform constant magnetic field. Suppose we choose the z axis to be in the direction of the field; that is, we shall write

$$\vec{B} = B\hat{k}$$

The differential equation of motion now reads

$$m\frac{d^2\vec{r}}{dt^2} = q(\vec{v} \times B\vec{k}) = qB\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & 1 \end{vmatrix}$$

$$m(\ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}) = qB(\dot{y}\hat{i} - \dot{x}\hat{j})$$

$$m\ddot{x} = qB\dot{y} \qquad m\ddot{y} = -qB\dot{x} \qquad \ddot{z} = 0$$

$$m\dot{x} = qBy + c_1 \qquad m\dot{y} = -qBx + c_2 \qquad \dot{z} = const. = \dot{z}_0$$

$$\omega = qB/m \qquad C_1 = c_1/m \qquad C_2 = c_2/m$$

$$\dot{x} = \omega y + C_1 \qquad \dot{y} = -\omega x + C_2 \qquad \dot{z} = \dot{z}_0$$

$$\ddot{x} = \omega \dot{y} \qquad \ddot{x} + \omega^2 x = \omega^2 a \qquad a\omega = C_2$$

$$x = a + A\cos(\omega t + \theta_0)$$

$$\dot{x} = -A\omega \sin(\omega t + \theta_0)$$

$$\omega y + C_1 = -A\omega \sin(\omega t + \theta_0)$$

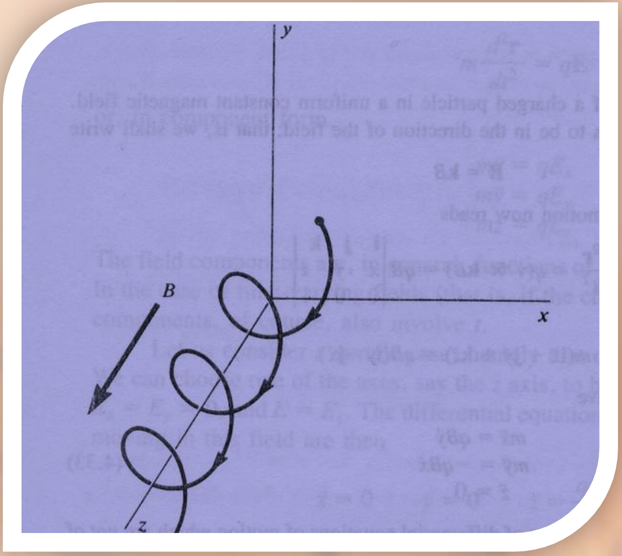
$$y = \frac{-A\omega \sin(\omega t + \theta_0) - C_1}{\omega} \qquad b = -C_1/\omega$$

$$y = b - A\sin(\omega t + \theta_0)$$

$$x = a + A\cos(\omega t + \theta_0)$$

$$\dot{z} = \dot{z}_0 = const. \qquad z = \dot{z}_0 t + z_0$$

$$(x-a)^2 + (y-b)^2 = A^2$$



$$\dot{y} = -A\omega\cos(\omega t + \theta_0)$$

$$\dot{x} = -A\omega\sin(\omega t + \theta_0)$$

$$\dot{x}^2 + \dot{y}^2 = A^2\omega^2 = A^2\left(\frac{qB}{m}\right)^2$$

$$v_1 = (\dot{x}^2 + \dot{y}^2)^{1/2} \qquad A = \frac{v_1}{\omega} = v_1\frac{m}{qB}$$