ANALYTICAL MECHANICS 1

Lecture 11

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General Motion of a Particle in Three Dimensions

$$\vec{F} = \frac{d\vec{p}}{dt} \qquad \vec{p} = m\vec{v}$$

$$F_x = \frac{dp_x}{dt}, \quad F_y = \frac{dp_y}{dt}, \quad F_z = \frac{dp_z}{dt}$$

$$\int_0^t \vec{F}(t)dt = \vec{p}(t) - \vec{p}(0)$$

$$= m\vec{v}(t) - m\vec{v}(0)$$

Angular Momentum

$$\vec{F} = \frac{d\vec{p}}{dt} \qquad \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt} \stackrel{?}{=} \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$\frac{d}{dt}(\vec{r} \times \vec{p}) = \vec{v} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$\vec{v} \times \vec{p} = \vec{v} \times m\vec{v} = m\vec{v} \times \vec{v} = 0$$

$$\vec{r} \times \vec{F} = \frac{d}{dt}(\vec{r} \times \vec{p}) \qquad \rightarrow \vec{\tau} = \frac{d\vec{L}}{dt} \qquad \vec{L} = \vec{r} \times \vec{p}$$

The Work Principle

$$\vec{F} = \frac{d\vec{p}}{dt} \qquad \vec{F}.\vec{v} = \frac{d\vec{p}}{dt}.\vec{v} = \frac{d(m\vec{v})}{dt}.\vec{v}$$

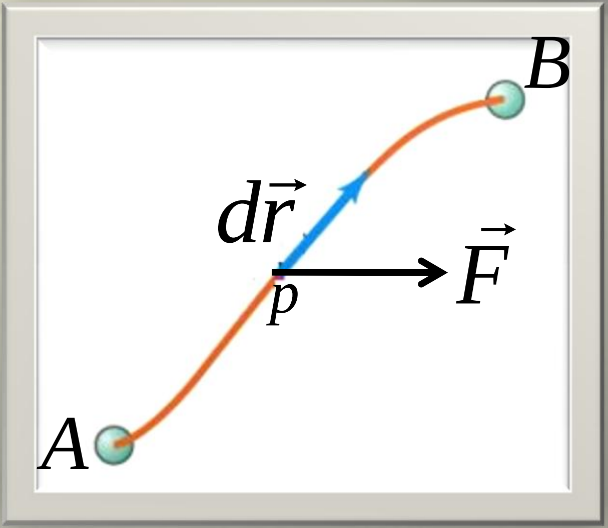
$$\frac{d(\vec{v}.\vec{v})}{dt} = 2\vec{v}.\frac{d\vec{v}}{dt} \qquad \vec{F}.\vec{v} = \frac{d}{dt}\left(\frac{1}{2}m\vec{v}.\vec{v}\right) = \frac{dT}{dt}$$

$$T = \frac{1}{2}mv^2 \qquad \int \vec{F}.\vec{v}dt = \int dT \qquad \vec{v}dt = d\vec{r}$$

$$\int \vec{F}.d\vec{r} = \int dT = \Delta T$$

The work done on the particle is equal to the increment in the kinetic energy

Conservation Forces and Force Fields



$$W = \int_A^B \vec{F} . d\vec{r}$$

The Potential Energy Function in Three Dimensional Motion

$$\vec{F}.d\vec{r} = -dV(\vec{r}) \qquad \vec{F}_x dx = -dV$$

$$\vec{F}.d\vec{r} = dT \qquad \Delta T = -\Delta V \qquad \Delta(T + V) = 0$$

$$\frac{1}{2}mv^2 + V(r) = E \qquad \vec{F} + \vec{F}'$$

$$dT = \vec{F}.d\vec{r} + \vec{F}'.d\vec{r} = -dV + \vec{F}'.d\vec{r}$$

$$d(T + V) = \vec{F}'.d\vec{r}$$

Del Operator ∇

Gradient and the Del Operator in Mechanics

$$\vec{F}.d\vec{r} = -dV$$

$$F_x dx + F_y dy + F_z dz = -\frac{\partial V}{\partial x}dx - \frac{\partial V}{\partial y}dy - \frac{\partial V}{\partial z}dz$$

$$F_x = -\frac{\partial V}{\partial x} \qquad F_y = -\frac{\partial V}{\partial y} \qquad F_z = -\frac{\partial V}{\partial z}$$

$$\vec{F} = -\frac{\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial z}\hat{k} = -\vec{\nabla}V$$

$$\vec{F} = -\vec{\nabla}V$$

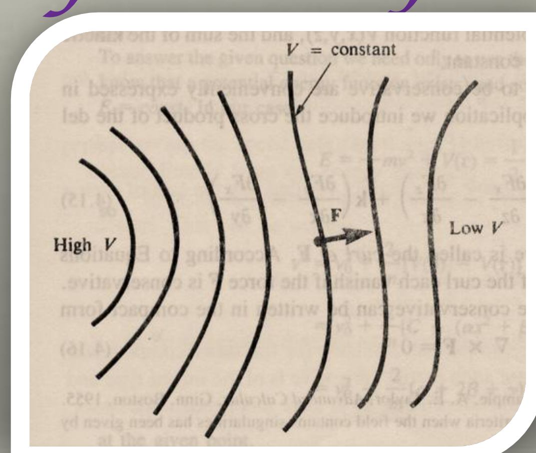
Del Operator ∇

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

"del"
"nabla"

The gradient of a function is a vector that represent the maximum spatial derivative of the function in direction and magnitude..

Notice that the result is a linear combination of components and basis vectors and therefore ***the gradient of a scalar function is a vector.***



Curl $\nabla\times$

Conditions for the Existence of a Potential Function

$$F_x = -\frac{\partial V}{\partial x} \qquad \frac{\partial F_x}{\partial y} = -\frac{\partial^2 V}{\partial y\,\partial x} \qquad \frac{\partial F_y}{\partial x} = -\frac{\partial^2 V}{\partial x\,\partial y}$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \qquad \frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x} \qquad \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}$$

$$\nabla\times\vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{j}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k}$$

Curl $\nabla\times$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{k} = 0$$

Physically the curl is a measure of the rotational properties of a vector about a point.

$$\nabla\times\mathcal{F}=0$$

Divergence $\nabla.$

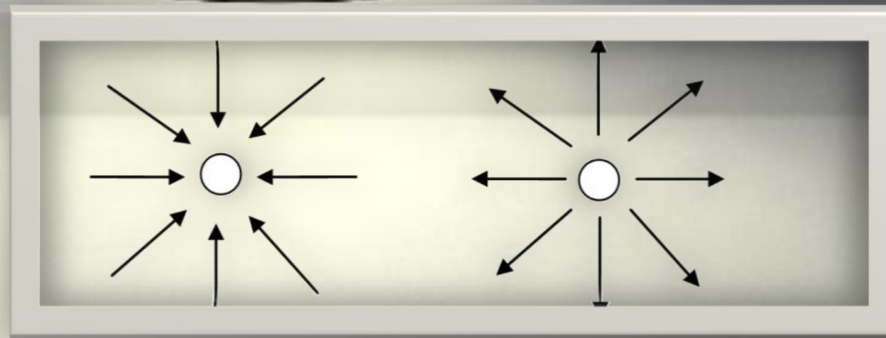
The divergence of the vector field.

$$\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot\left(V_x\hat{i}+V_y\hat{j}+V_z\hat{k}\right)$$

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}$$ (a scalar field!)

From a physical standpoint, the divergence is a measure of the addition or removal of a vector quantity. A system with positive divergence is called a source. A system with negative divergence is called a sink.

Divergence $\nabla .$



Example(1): Find the force function.

$$V(x) = \alpha x^2 + \beta xy + \gamma z + C$$

$$\alpha, \beta, \gamma, C \quad are \quad \text{cost}.$$

$$F = -\nabla V = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right)$$

$$-(2x\alpha + y\beta)\hat{i} - (x\beta)\hat{j} - \gamma\hat{k}$$

Example(2):

$$F = -(2x\alpha + y\beta)\hat{i} - (x\beta)\hat{j} - \gamma\hat{k}$$

$$T + V = E = const. \qquad \vec{r} = \hat{i} + 2\hat{j} + \hat{k}$$

$$E = \frac{1}{2}mv^2 + V(r) = \frac{1}{2}mv_0^2 + V(0)$$

$$v^2 = v_0^2 + \frac{2}{m}\left[V(0) - V(r)\right]$$

$$v^2 = v_0^2 + \frac{2}{m}\left[C - (\alpha x^2 + \beta xy + \gamma z + C)\right]$$

$$= v_0^2 - \frac{2}{m}(\alpha + 2\beta + \gamma)$$

Example(3):

$$\vec{F} = xy\hat{i} + xz\hat{j} + yz\hat{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & xz & yz \end{vmatrix} = (z - x)\hat{i} + 0\hat{j} + (z - x)\hat{k}$$

Example(4):

$$\vec{F} = (ax + by^2)\hat{i} + cxy\hat{j}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ ax + by^2 & cxy & 0 \end{vmatrix} = (c - 2b)y\hat{k}$$

Example(5):

$$F_r = (-k/r^2)\hat{e}_r$$

$$\nabla \times \vec{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & \hat{e}_\theta r & \hat{e}_\varphi r\sin\theta \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \varphi} \\ F_r & rF_\theta & rF_\varphi \sin\theta \end{vmatrix}$$

$$F_r = -k/r^2, \quad F_\theta = 0, \quad F_\varphi = 0$$

$$\nabla \times \vec{F} = \frac{\hat{e}_\theta}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{-k}{r^2}\right) - \frac{\hat{e}_\varphi}{r}\frac{\partial}{\partial\theta}\left(\frac{-k}{r^2}\right) = 0$$

Forces of the Separable Type. Projectile Motion

$$\vec{F} = F_x(x)\hat{i} + F_y(y)\hat{j} + F_z(z)\hat{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x(x) & F_y(y) & F_z(z) \end{vmatrix} = 0 \qquad \frac{\partial F_z(z)}{\partial y} - \frac{\partial F_y(y)}{\partial z}$$

$$m\ddot{x} = F_x(x), \; m\ddot{y} = F_y(y), \; m\ddot{z} = F_z(z)$$

Motion of a Projectile in a Uniform Gravitational Field

a) No Air Resistance

$$\vec{F} = F_x(x)\hat{i} + F_y(y)\hat{j} + F_z(z)\hat{k} = 0\hat{i} + 0\hat{j} - mg\hat{k}$$

$$m\frac{d^2\vec{r}}{dt^2} = -mg\hat{k}$$

$$\frac{1}{2}mv^2 + V(r) = E$$

$$\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz = \frac{1}{2}mv_0^{\,2}$$

$$v^2 = v_0^{\,2} - 2gz$$

$$\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = -g\hat{k}$$

$$\frac{d\vec{r}}{dt} = -gt\hat{k} + \vec{v}_0$$

$$\vec{r} = -\frac{1}{2}gt^2\hat{k} + \vec{v}_0 t + \vec{r}_0 \qquad \vec{r}_0 = 0$$

$$x = \dot{x}_0 t \qquad y = \dot{y}_0 t \qquad z = \dot{z}_0 t - \frac{1}{2}gt^2$$

$$y = bx \qquad b = \frac{\dot{y}_0}{\dot{x}_0} \qquad \dot{y}_0 = 0$$

$$x = \dot{x}_0 t \qquad z = \dot{z}_0 t - \frac{1}{2} g t^2$$

$$z = \frac{\dot{z}_0}{\dot{x}_0} x - \frac{g}{2\dot{x}_0^2} x^2$$

$$a = \dot{z}_0 / \dot{x}_0 \ , \ b = g / 2\dot{x}_0^2$$

$$z = ax - bx^2$$