ANALYTICAL MECHANICS 1

Lecture 10

Sahraei

Physics Department,

Razi University

http://www.razi.ac.ir/sahraei

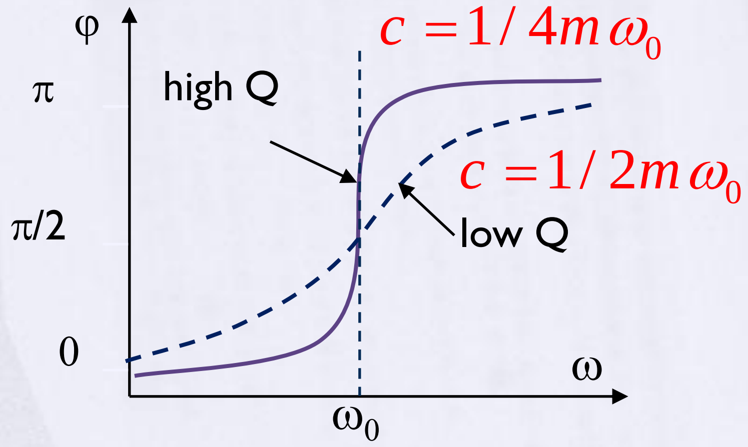
Phase Angle

$$tg\,\varphi = \frac{\omega c}{k - m\omega^2}$$

$$\gamma = c / 2m \qquad \omega_0 = (k / m)^{1/2}$$

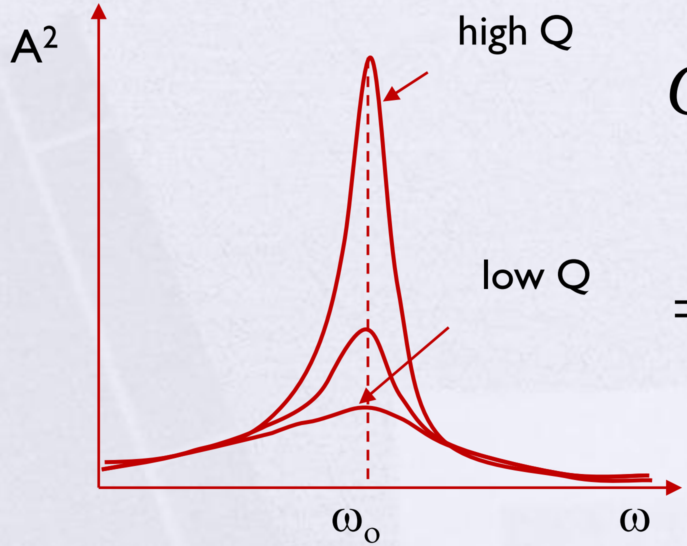
$$tg\,\varphi = \frac{2\gamma\omega}{\omega_0^{\,2} - \omega^2}$$

$$\varphi = tg^{-1}\frac{2\gamma\omega}{\omega_0^{\,2} - \omega^2}$$



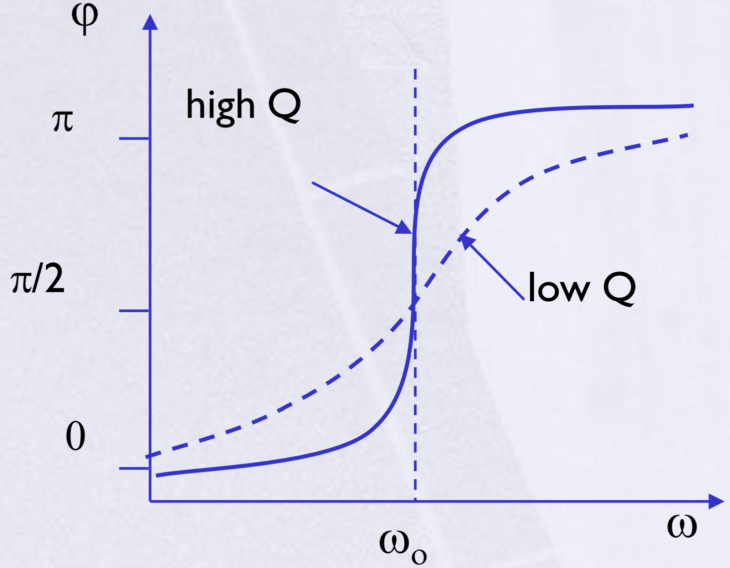
High Q = sharp resonance
Damping reduces Q

The *width* of the resonance curves depends on ***c***, i.e., on the amount of damping



$$Q = \pi \frac{\text{decay time}}{\text{period}} = \pi \frac{1/\gamma}{2\pi/\omega_o} = \frac{\omega_o}{2\gamma}.$$

Bandwidth, BW = difference between the two half-power frequencies



The higher the Q, the smaller the bandwidth

Velocity Resonance

$$x = A\cos(\omega t - \varphi) \qquad \dot{x} = -\omega A \sin(\omega t - \varphi)$$

$$v(\omega) = \frac{\omega F_0 / m}{\sqrt{(\omega_0^2 - \omega^2) + 4\gamma^2 \omega_0^2}} \qquad \gamma > \frac{\omega_0}{\sqrt{2}}$$

Example: *The exponential damping factor* γ *of a spring suspension system is one-tenth the critical value. If the undamped frequency is* ω_0*, find (a) the resonant frequency, (b) the quality factor, (c) the phase angle* φ *when the system is driven at a frequency* $\omega=\omega_0/2$*, and (d)the steady-state amplitude at this frequency.*

$$a)\gamma = \gamma_{crit} / 10 = \omega_0 / 10 \qquad \omega_r^{\ 2} = \omega_d^{\ 2} - \gamma^2$$

$$\omega_r = \sqrt{\omega_0^{\ 2} - 2(\omega_0 / 10)^2} = \omega_0 \sqrt{0.98} = 0.99\omega_0$$

$$b)Q \simeq \frac{\omega_0}{2\gamma} = \frac{\omega_0}{2(\omega_0 / 10)} = 5$$

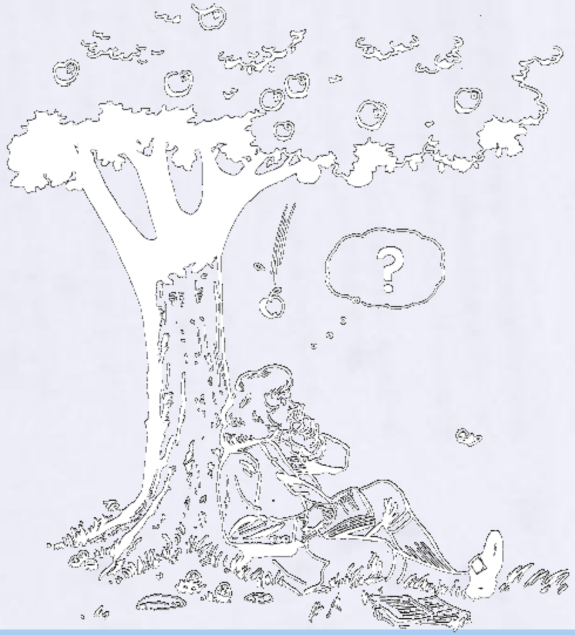
$$c)\varphi = tg^{-1} \frac{2\gamma\omega}{\omega_0^{\ 2} - \omega^2} = \tan^{-1}\left[\frac{2(\omega_0 / 10)(\omega_0 / 2)}{\omega_0^{\ 2} - (\omega_0 / 2)^2}\right] = 7.6^\circ$$

$$d)A(\omega) = \frac{F_0 / m}{\left[(\omega_0^{\ 2} - \omega^2)^2 + 4\gamma^2\omega^2\right]^{1/2}} \qquad \omega = \omega_0 / 2$$

$$A(\omega)=\frac{F_0/m}{\left[(\omega_0^{\,2}-\omega_0^{\,2}/4)^2+4(\omega_0/10)^2(\omega_0/2)^2\right]^{1/2}}$$

$$A(\omega)=\frac{F_0/m}{0.7506\omega_0^{\,2}}=1.332\frac{F_0}{m\omega_0^{\,2}}$$

$$\frac{F_0}{m\omega_0^{\,2}}=\frac{F_0}{k}$$



The Nonlinear Oscillator. Method of Successive Approximations

$$F(x) = -kx$$ Linear Oscillator

$$F(x) = -kx + \varepsilon(x)$$ Nonlinear Oscillator

$$\varepsilon(x) = \varepsilon_2 x^2 + \varepsilon_3 x^3 + \dots$$

$$m\ddot{x} + kx = \varepsilon_2 x^2 + \varepsilon_3 x^3 + \dots \qquad m\ddot{x} + kx = \varepsilon_3 x^3$$

$$\ddot{x} + \frac{k}{m} x = \frac{\varepsilon_3}{m} x^3 \qquad \ddot{x} + \omega_0^2 x = \lambda x^3$$

We shall find the solution by the method of successive approximations.

$$\lambda = 0 \rightarrow x = A\cos\omega_0 t$$

Suppose we try a first approximation of the same form

$$\lambda \neq 0 \rightarrow x = A\cos\omega t \qquad \omega \neq \omega_0$$

$$-A\omega^2\cos\omega t + A\omega_0^{\,2}\cos\omega t = \lambda A^3\cos^3\omega t$$

$$= \lambda A^3\left(\frac{3}{4}\cos\omega t + \frac{1}{4}\cos 3\omega t\right)$$

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

$$\left(-\omega^2 + \omega_0^{\,2} - \frac{3}{4}\lambda A^2\right)A\cos\omega t - \frac{1}{4}\lambda A^3\cos 3\omega t = 0$$

$$-\omega^2+\omega_0{}^2-\frac{3}{4}\lambda A^2=0 \qquad \omega^2=\omega_0{}^2-\frac{3}{4}\lambda A^2$$

$$\omega=f(A) \qquad \omega=\sqrt{\frac{k}{m}}$$

$$x=A\cos\omega t+B\cos 3\omega t$$ *Second trail solution*

$$(-\omega^2+\omega_0{}^2-\frac{3}{4}\lambda A^2)A\cos\omega t+(-9B\omega^2+\omega_0{}^2B-\frac{1}{4}\lambda A^3)\cos 3\omega t)$$
$$+(\text{terms involving } B\lambda \text{ and higher multiples of } \omega t)=0$$

$$-9B\omega^2+\omega_0{}^2B-\frac{1}{4}\lambda A^3=0 \qquad B=\frac{\frac{1}{4}\lambda A^3}{-9\omega^2+\omega_0{}^2}$$

$$B=\frac{\frac{1}{4}\lambda A^{3}}{-9\omega^{2}+\omega_{0}{}^{2}}=\frac{\lambda A^{3}}{-32\omega_{0}{}^{2}+27\lambda A^{2}}\simeq-\frac{\lambda A^{3}}{32\omega_{0}{}^{2}}$$

$$x=A\cos\omega t-\frac{\lambda A^{3}}{32\omega_{0}{}^{2}}\cos 3\omega t$$

$$\omega=\omega_{0}(1-\frac{3\lambda A^{2}}{4\omega_{0}{}^{2}})^{1/2}$$

Example: The simple pendulum as a nonlinear oscillator

$$\ddot{\theta} + (g/l)\sin\theta = 0$$

$$\sin\theta \simeq \theta$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}$$

$$\ddot{\theta} + \frac{g}{l}\theta = \frac{g\theta^3}{3!l}$$

$$\frac{g}{l} = \omega_0^{\ 2}$$

$$\ddot{\theta} + \omega_0^{\ 2}\theta = \frac{\omega_0^{\ 2}}{3!}\theta^3$$

$$\ddot{\theta} + \omega_0^{\ 2}\theta = \lambda\theta^3$$

$$\lambda = \frac{\omega_0^{\ 2}}{3!} = \frac{\omega_0^{\ 2}}{6}$$

$$\omega = \omega_0 \left(1 - \frac{3\lambda A^2}{4\omega_0^2}\right)^{1/2}$$

$$\omega = \omega_0 \left[1 - \frac{3\left(\frac{\omega_0^2}{6}\right)A^2}{4\omega_0^2}\right]^{1/2} = \omega_0 \left(1 - \frac{A^2}{8}\right)^{1/2}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{l/g}\left(1 - \frac{A^2}{8}\right)^{-1/2} = T_0\left(1 - \frac{A^2}{8}\right)^{-1/2}$$

$$A = \pi/2 \ rad \qquad T = T_0\left(1 - \frac{\pi^2}{32}\right)^{-1/2} = 1.2025T_0$$

2)

$$x(t) = A\sin(\omega t - \varphi)$$

$$v(t) = A\omega\cos(\omega t - \varphi)$$

The system passes the center at the moment t_i defined by the condition

$$x(t_i) = A\sin(\omega t_i - \varphi) = 0 \qquad \omega t_i + \varphi = \pi i$$

where $i = 0, 1, 2, 3, \ldots$ Velocity at these moments equals

$$v(t_i) = A\omega\cos(\omega t_i - \varphi) = A\omega\cos(\pi i) = \pm A\omega$$

$$\omega = \frac{|v(t_i)|}{A}$$

$$T = \frac{2\pi A}{|v(t_i)|}$$

$$T = 2\pi/5 \approx 1.26 \quad s$$

3) $f = 10Hz,\quad t = 0,\quad x_0 = 0.25m,\quad v_0 = 0.1m/s$

$$x(t) = x_0 \cos\omega t + \frac{v_0}{\omega}\sin\omega t$$

$$x(t) = x_0\cos(2\pi f t) + \frac{v_0}{2\pi f}\sin(2\pi f t)$$

$$x(t) = 0.25\cos(20\pi t) + \frac{0.005}{\pi}\sin(20\pi t)$$

5)

$$x(t) = A\sin(\omega t - \varphi)$$

$$v(t) = A\omega\cos(\omega t - \varphi)$$

$$v(t) = A\omega\sqrt{1 - \sin^2(\omega t - \varphi)}$$

$$= \omega\sqrt{A^2 - A^2\sin^2(\omega t - \varphi)} \quad = \omega\sqrt{A^2 - x^2}$$

$$\dot{x}_1 = \omega\sqrt{A^2 - x_1^2} \qquad \dot{x}_2 = \omega\sqrt{A^2 - x_2^2}$$

$$A^2 = \frac{x_1^2\dot{x}_2^2 - x_2^2\dot{x}_1^2}{\dot{x}_2^2 - \dot{x}_1^2} \qquad \omega^2 = \frac{\dot{x}_1^2 - \dot{x}_2^2}{x_2^2 - x_1^2}$$

9) $$x(t) = Ae^{-\gamma t}\cos(\omega_d t - \varphi)$$

The moments t_i $(i = 1, 2, 3, ...)$ when $x(t)$ has maximums can be determined from the equation

$$\frac{dx}{dt} = -Ae^{-\gamma t_i}\left[\omega_d \sin(\omega_d t_i - \varphi) + \gamma\cos(\omega_d t_i - \varphi)\right] = 0$$

$$\tan(\omega_d t - \varphi) = -\gamma / \omega_d$$

$$\omega_d t_i + \varphi = \tan^{-1}(-\gamma / \omega_d) + 2\pi i$$

Therefore the time between two successive maxima is

$$t_{i+1} - t_i = \frac{2\pi}{\omega_d}$$

The amplitude of the ith maximum is

$$x(t_i) = Ae^{-\gamma t_i}\cos(\omega_d t_i - \varphi)$$

The amplitude of the $(i+1)$th maximum is

$$x(t_{i+1}) = Ae^{-\gamma t_{i+1}}\cos(\omega_d t_{i+1} - \varphi)$$

Since $t_{i+1} = t_i + \frac{2\pi}{\omega_d}$,

$$x(t_{i+1}) = e^{-\gamma 2\pi/\omega_d}\, x(t_i)$$

$$\frac{x(t_{i+1})}{x(t_i)} = e^{-2\pi\gamma/\omega_d})$$

the ratio $x(t_{i+1})/x(t_i)$ does not depend on time and is given by

11)

$$x(t) = e^{-\gamma t} \cos \omega_d t$$

After n cycles the amplitude drop is $1/e$, so therefore the time passed is $T = 1/\gamma$.

the time of n oscillation is determined by the condition

$$\omega_d T = 2\pi n$$

$$\frac{1}{\gamma} = \frac{2\pi n}{\omega_d} \qquad \omega_d = \sqrt{\omega_0^{\ 2} - \gamma^2} \qquad \gamma^2 = \frac{\omega_0^{\ 2}}{1 + 4\pi^2 n^2}$$

$$\omega_d = \omega_0 \sqrt{1 - \frac{1}{1 + 4\pi^2 n^2}} \qquad T_d / T = \omega_0 / \omega_d = \sqrt{1 + \frac{1}{4\pi^2 n^2}}$$

13)

$$A(\omega) = \frac{\gamma A_{\max}}{\sqrt{(\omega_0 - \omega)^2 + \gamma^2}}$$

$$\frac{A}{A_{\max}} = \frac{1}{2}$$

$$\omega_0 - \omega = \sqrt{3}\gamma$$

Some concepts for oscillations

restoring force: A force causes the system to return to some equilibrium state periodically and repeat the motion

natural frequency: Resonant oscillation period, determined by physics of the system alone. Disturb system to start, then let it go. Examples: pendulum clock, violin string

undamped oscillations: Idealized case, no energy lost, motion persists forever Example: orbit of electrons in atoms and molecules

damped oscillations: Oscillation dies away due to loss of energy, converted to heat or another form. Example: a swing eventually stops

simple harmonic oscillation: Undamped natural oscillation with $F = -kx$ (Hooke's Law); i.e. restoring force is proportional to the displacement away from the equilibrium state

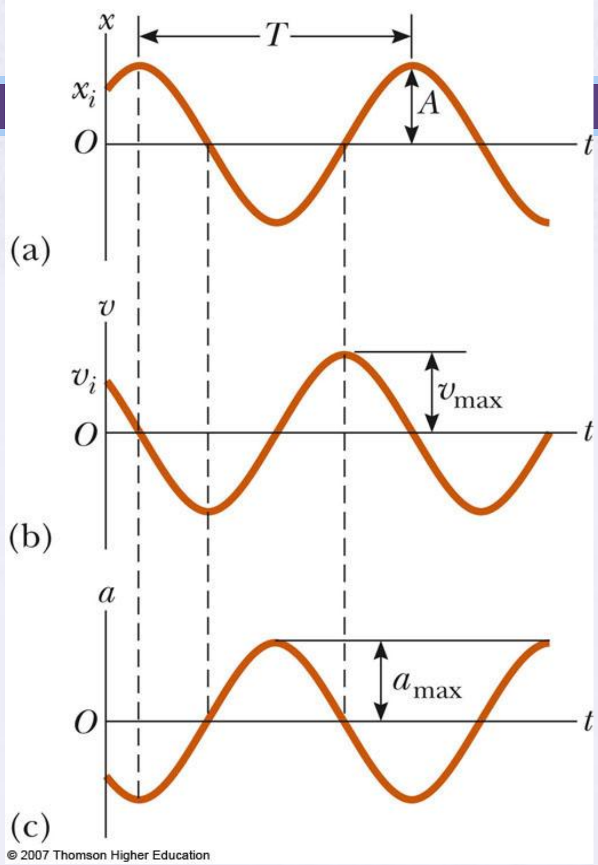
forced oscillations: External periodic force drives the system motion at it's own frequency/period, may not be the resonant frequency

Forced oscillations and resonance

- Swinging without outside help – free oscillations
- Swinging with outside help – forced oscillations
- If ω_d is a frequency of a driving force, then forced oscillations can be described by:

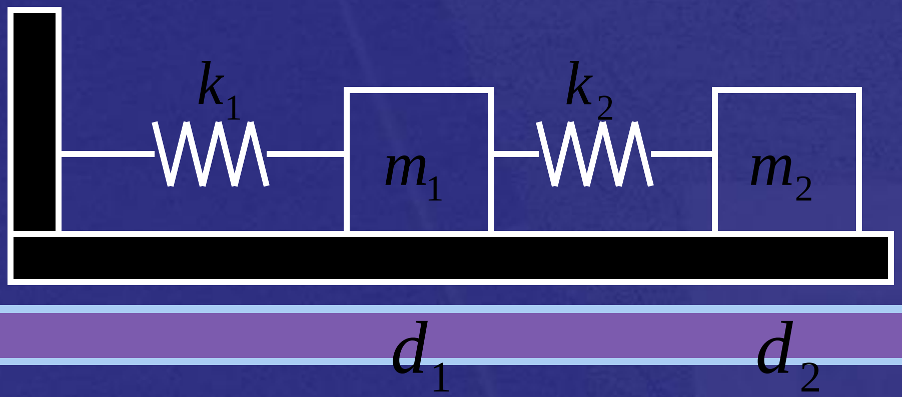
Graphs

- The graphs show:
  - (a) displacement as a function of time
  - (b) velocity as a function of time
  - (c) acceleration as a function of time
- The velocity is 90° out of phase with the displacement and the acceleration is 180° out of phase with the displacement



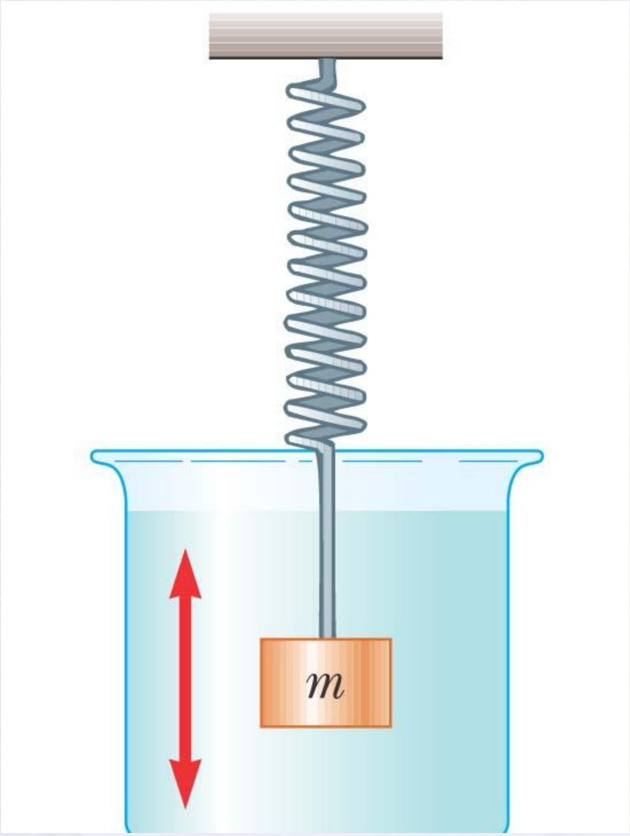
Writing the Equations of Motion: Example 3

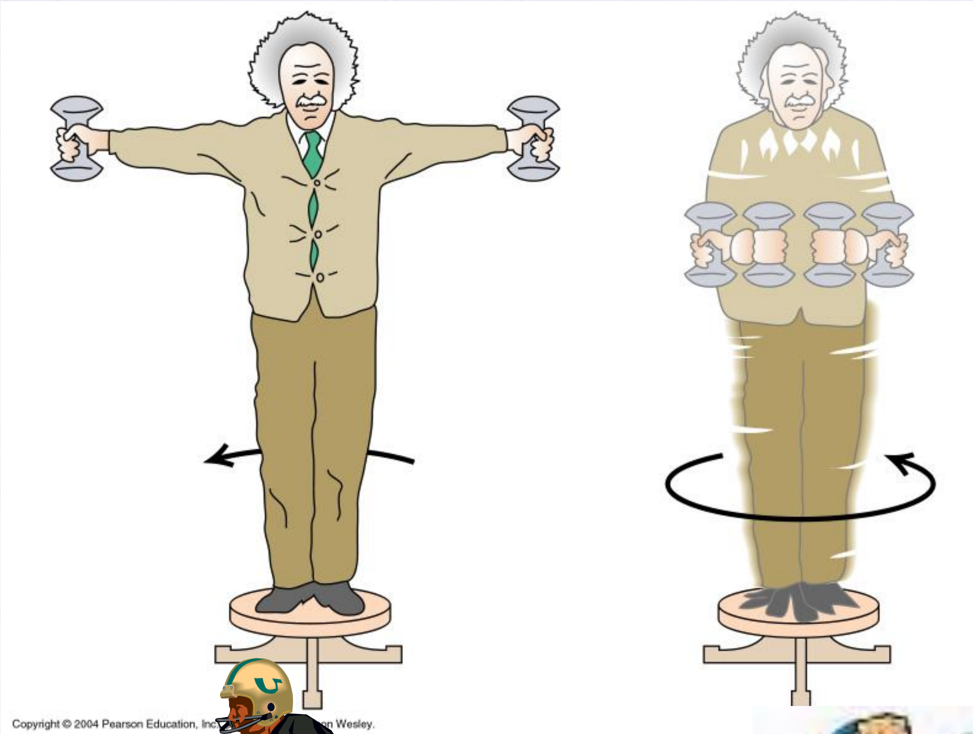
Write the equations of motion for the following:

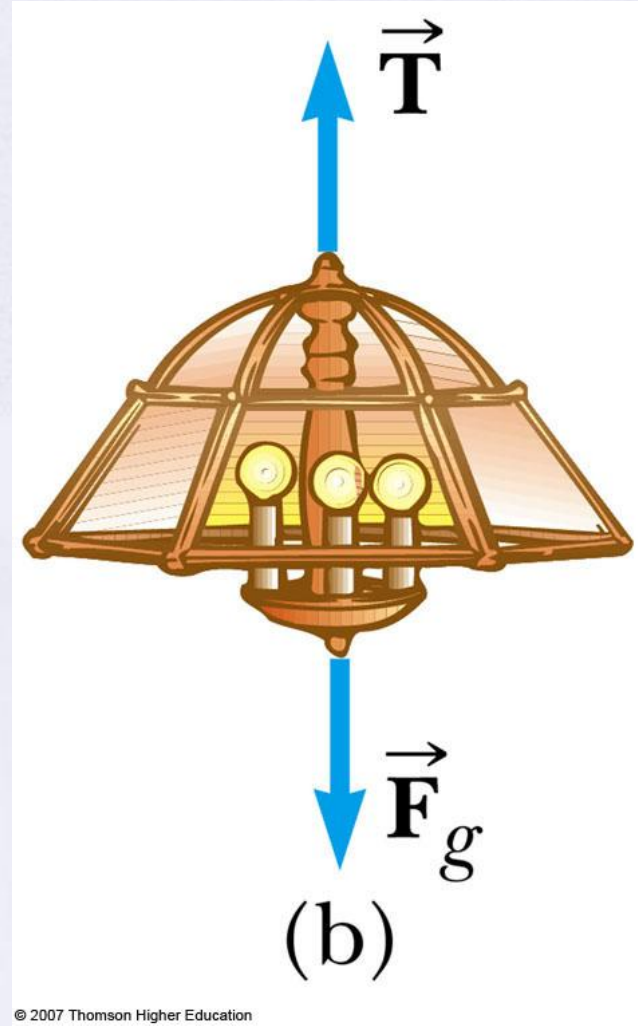


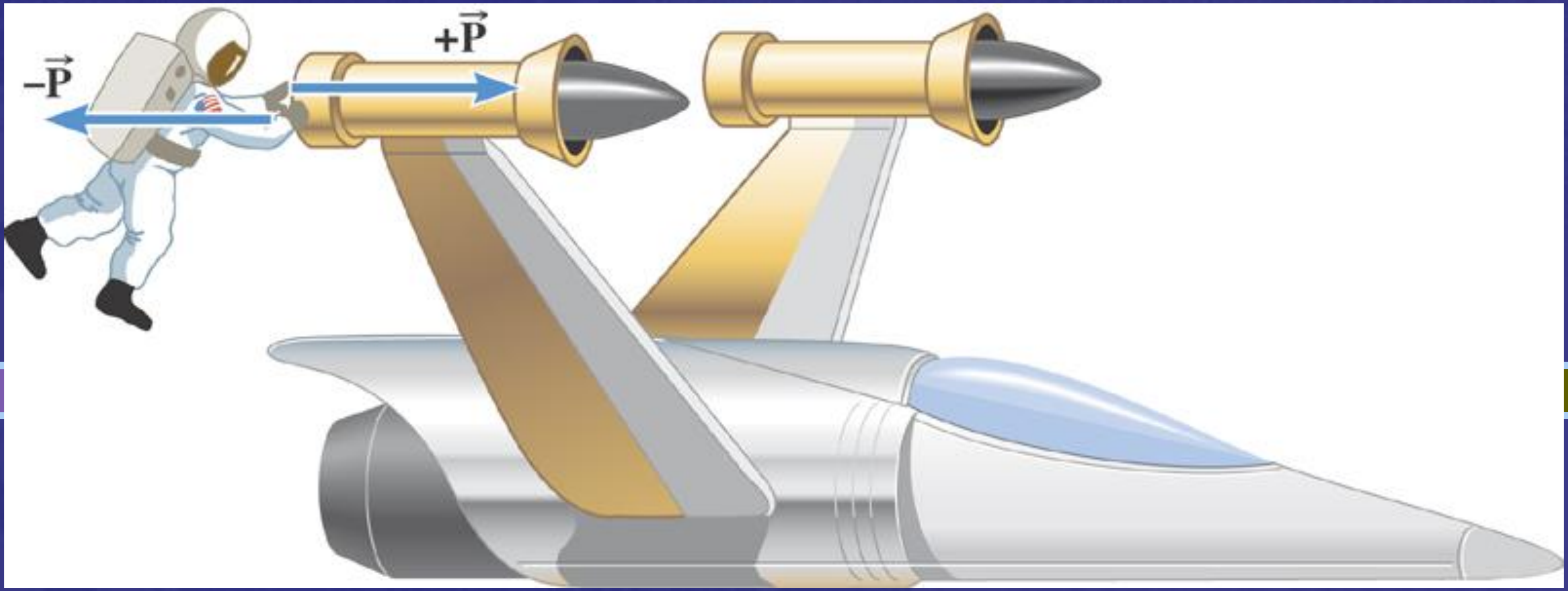
$$0 = -m_1\ddot{x}_1 - d_1\dot{x}_1 - k_1x_1 + k_2\left(x_2 - x_1\right)$$

$$0 = -m_2\ddot{x}_2 - d_2\dot{x}_2 + k_2\left(x_1 - x_2\right)$$


m


Copyright © 2004 Pearson Education, Inc
n Wesley.


$\vec{T}$
$\vec{F}_g$
(b)
© 2007 Thomson Higher Education


$-\vec{P}$
$+\vec{P}$

$$x(t) = Ae^{i(\omega t - \varphi)}$$

$$\dot{x}(t) = i\omega Ae^{i(\omega t - \varphi)} \qquad \ddot{x}(t) = -\omega^2 Ae^{i(\omega t - \varphi)}$$

$$-m\omega^2 Ae^{i(\omega t - \varphi)} + ic\omega Ae^{i(\omega t - \varphi)} + kAe^{i(\omega t - \varphi)} = F_0 e^{i\omega t}$$

$$-m\omega^2 A + ic\omega A + kA = F_0 e^{i\varphi} = F_0(\cos\varphi + i\sin\varphi)$$

$$A(k - m\omega^2) = F_0\cos\varphi \qquad c\omega A = F_0\sin\varphi$$

$$A^2(k - m\omega^2)^2 + c^2\omega^2 A^2 = F_0^2$$

SHM is the projection of uniform circular motion on a line in the plane of the motion. One period of SHM can be divided into 360^0.

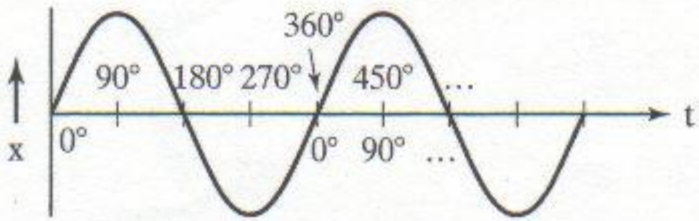


Figure 1-9 Phase of SHM curve.

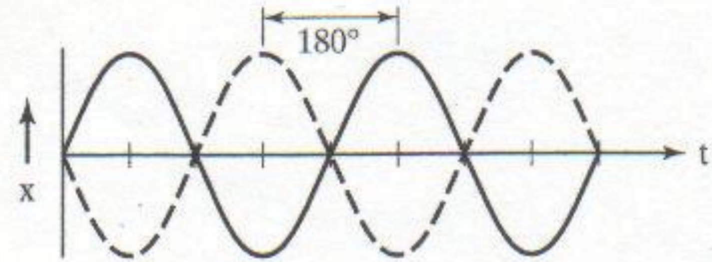


Figure 1-10 Two SHM curves differing in phase by 180° (out of phase).

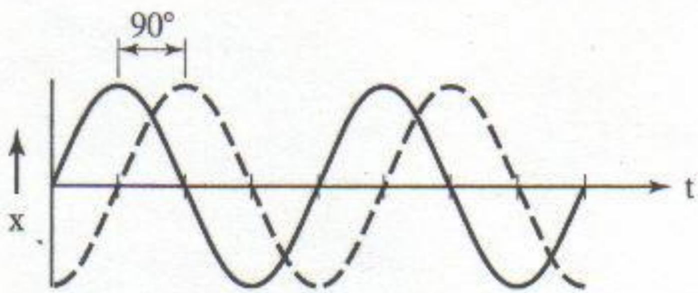


Figure 1-11 Solid curve 90° ahead of dashed curve.

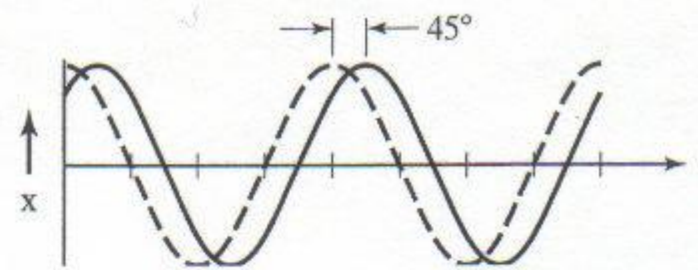


Figure 1-12 Solid curve 45° behind dashed curve.