Atmospheric Physics

Lecture 17

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$$\omega_p = \left(\frac{ne^2}{\varepsilon_0 m}\right)^{1/2} \qquad \omega_p = 2\pi f_p$$

$$f_p{}^2 = \frac{ne^2}{4\pi^2 \varepsilon_0 m}$$

n electron number density

The plasma frequency is the natural frequency of oscillation of electrons in a plasma displaced relative to the ion background.

The electron plasma frequency is one of a number of characteristic frequencies of a plasma

Since the charged particles in a plasma respond to static and oscillatory electromagnetic fields, strong interactions can occur between these plasma waves and the underlying charged particles in the plasma.

These strong interactions are often called instabilities.

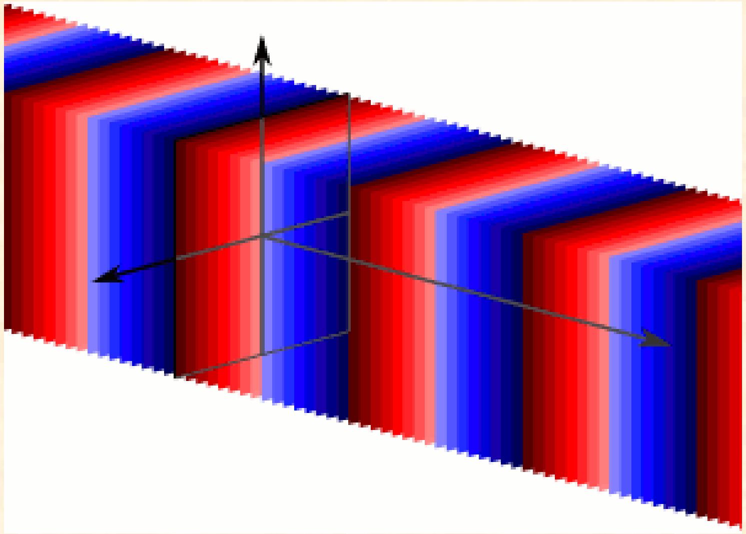
Electron plasma oscillations at the plasma frequency are one example of an instability in a plasma.

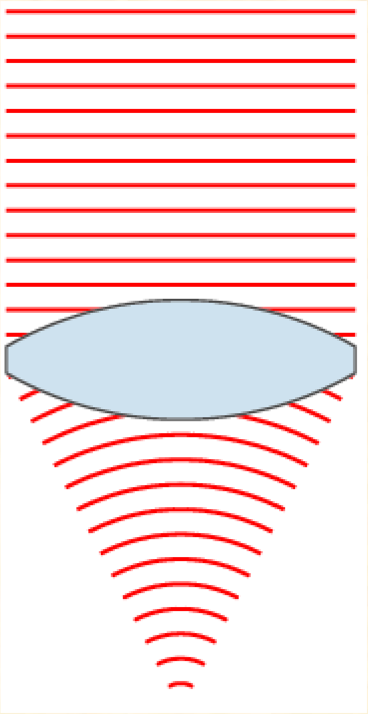
In many cases, plasma waves and instabilities are important in understanding the state of the plasma,

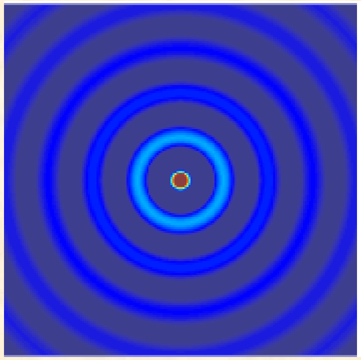
such waves do not propagate very far in a plasma and are strongly influenced by the magnetized plasma when they do propagate.

Phase and Group Velocity

The velocity of a wave can be defined in many different ways, partly because there many different kinds of waves, and partly because we can focus on different aspects or components of any given wave.



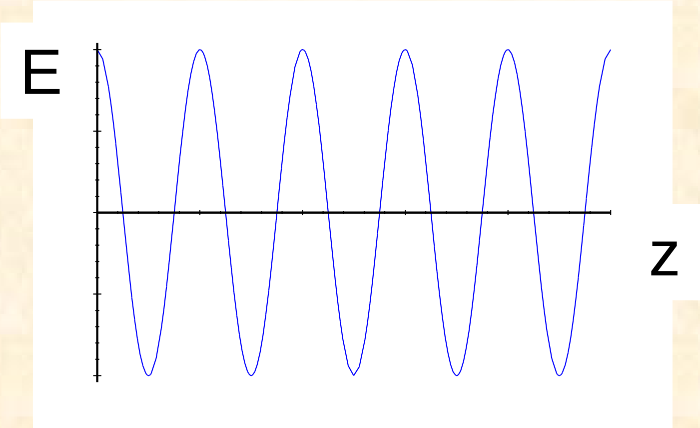




Here, plane wavefronts become spherical after going through the lens.

Propagating Electromagnetic Waves: Phase Velocity

monochromatic plane wave



phase velocity

$$\upsilon_p = \frac{\Delta z}{\Delta t} = \frac{\omega}{k}$$

$$E(z,t) = Ae^{i(kz-\omega t)}$$

phase $\quad \phi = kz - \omega t$

Points of constant phase move a distance Δz in a time Δt

Relationship between wavelength and frequency in free space:

$$\lambda = \frac{c}{f}$$

Relationship between wavelength and frequency in a material medium:

$$\boxed{\lambda = \frac{v_p}{f}}$$

*The **group velocity** of a wave is the velocity with which the variations in the shape of the wave's amplitude propagates through space. The group velocity is defined by the equation:*

$$v_g \equiv \frac{\partial \omega}{\partial k}$$

where:
v_g is the group velocity
ω is the wave's angular frequency
k is the wave number

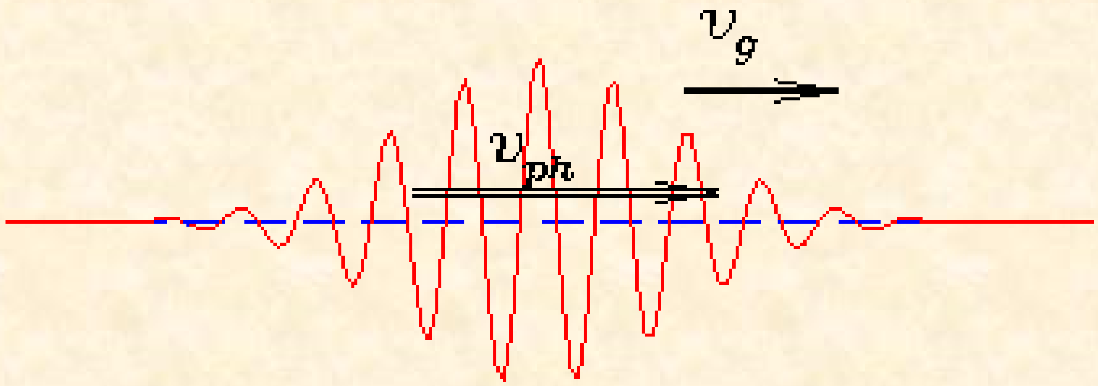
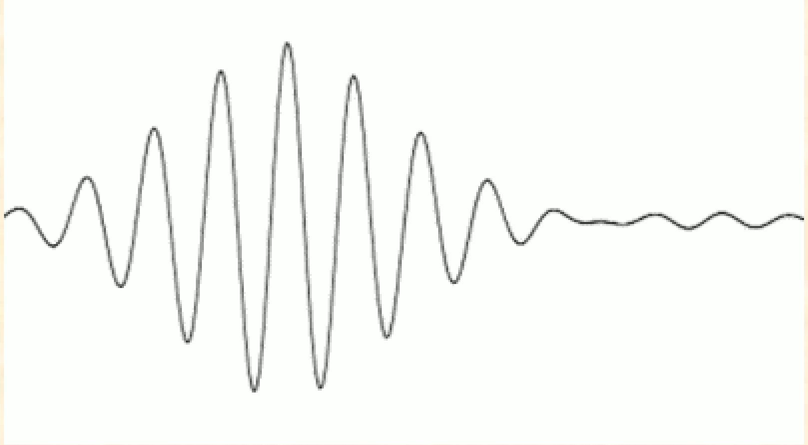
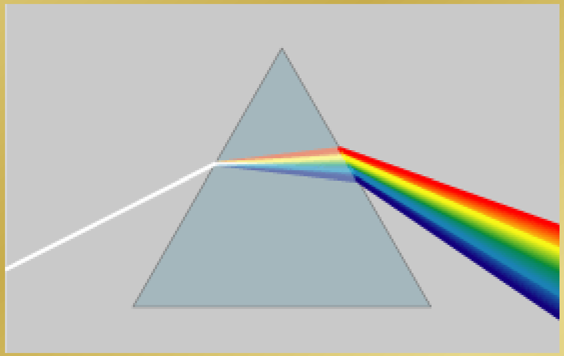


Figure: ***A WAVE PACKET moving at*** v_g ***with ``wavelets'' moving through it at*** v_{ph} **.**





*In optics, **dispersion** is a phenomenon that causes the separation of a wave into spectral components with different frequencies, due to a dependence of the wave's speed on its frequency.*

If velocity of propagation is independent of freq i.e. that is there is no dispersion then all component of the pulse travel with same velocity so the velocity of the resultant is the same as the phase velocity.

In a nondispersive wave medium, waves can propagate without deformation.

*In **non dispersive medium**, phase velocity is independent of wavelength i.e. dv/dk=0*

$$v_g = v_p$$

*In **dispersive medium**, each component of pulse has its own velocity of propagation so the velocity of the pulse is not the same as the phase velocity.*

*In a **dispersive medium** the **group velocity** is the velocity with which a signal is transmitted.*

$$v_g \neq v_p$$

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}.\vec{r} - \omega t)}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$E(r, t = 0) = E_0 e^{i\hat{k}.\vec{r}}$$

$$\vec{B} = \mu_0 H$$

$$\nabla \times \vec{E} = ikE_0 e^{ikr}$$

$$-\nabla \times \nabla \times \vec{E} = \nabla \times \frac{\partial \vec{B}}{\partial t} = \nabla \times \frac{\partial}{\partial t} \mu_0 \vec{H} =$$

$$= \mu_0 \frac{\partial}{\partial t} \nabla \times \vec{H} = \mu_0 \frac{\partial}{\partial t} (\vec{J} + \frac{\partial D}{\partial t}) =$$

$$\vec{J} = g\vec{E}$$

$$D = \varepsilon_0 \vec{E}$$

$$= \mu_0 \frac{\partial}{\partial t}(g\vec{E} + \varepsilon_0 \frac{\partial E}{\partial t}) =$$

$$= \mu_0 g \frac{\partial \vec{E}}{\partial t} + \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} =$$

$$= \mu_0 g(-i\omega)\vec{E} + \mu_0 \varepsilon_0 (\omega^2)\vec{E} =$$

$$= (-\mu_0 g i\omega + \mu_0 \varepsilon_0 \omega^2)\vec{E}$$

$$-\nabla\times\nabla\times\vec{E} = -\vec{k}\times(\vec{k}\times\vec{E}) = -(\vec{k}.\vec{E})\vec{k} + k^2\vec{E}$$

$$k^2 = (-\mu_0 g i\omega + \mu_0\varepsilon_0\omega^2)$$

$$k^2 = \mu_0\frac{g_0}{1-i\omega\tau}\omega + \mu_0\varepsilon_0\omega^2 = -\mu_0\frac{ne^2\tau/m}{1-i\omega\tau}\omega + \mu_0\varepsilon_0\omega^2$$

$$k^2 = -\frac{\mu_0\varepsilon_0\omega_p{}^2\tau\omega}{1-i\omega\tau} + \mu_0\varepsilon_0\omega^2$$

$$\Rightarrow k^2c^2 = -\frac{\omega_p{}^2\tau\omega}{1-i\omega\tau} + \omega^2$$

$$k^2c^2(1-i\omega\tau) = -\omega_p{}^2\tau\omega + \omega^2(1-i\omega\tau)$$

$$\begin{cases} k^2c^2 = -\omega_p{}^2 + \omega^2 \\ -k^2c^2 i\omega\tau = -i\omega^2\omega\tau \rightarrow \omega^2 = k^2c^2 \end{cases}$$

$$\omega^2 = k^2c^2 + \omega_p^2$$

Electromagnetic waves propagating through an unmagnetised plasma obey the dispersion relation

where k is the wave number. Consequently, waves with frequency below the plasma frequency cannot propagate in the plasma. As a corollary, waves with $\omega < \omega_p$ *incident on the plasma from outside are reflected .*

This is the effect that causes short wave radio signals to bounce off the ionosphere.

$$\begin{cases} n = c/v_p = \dfrac{c}{\omega/k} = \dfrac{kc}{\omega} \\ v_p = \omega/k \end{cases}$$

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$

$$n^2 = 1 - \frac{\omega_p^2}{\omega^2} = 1 - \frac{f_p^2}{f^2}$$

$$\text{if} \quad f > f_p \Rightarrow n < 1 \quad \text{and real}$$

$$\text{if} \quad f < f_p \Rightarrow \text{imaginary}$$

Dielectric constant of a plasma

A plasma is very similar to a gaseous medium, expect that the electrons are *free*: *i.e.*, there is no restoring force due to nearby atomic nuclii.

$$\begin{cases} v_p = \dfrac{c}{n} = \dfrac{c}{\sqrt{\varepsilon}} \\ v_p = \dfrac{\omega}{k} \end{cases}$$

$$\rightarrow \omega^2 = \frac{k^2 c^2}{\varepsilon}$$

$$\omega^2 = k^2 c^2 + \omega_p^2$$

$$\rightarrow 1 - \frac{\omega_p^2}{\omega^2} = \frac{k^2 c^2}{\omega^2}$$

$$= \varepsilon = \frac{c^2}{v_p^{\,2}}$$

$$v_p = \frac{c}{\sqrt{1 - \omega^2{}_p / \omega^2}}$$

How should we interpret this?

Consider an infinite plane-wave, of frequency, ω , greater than the plasma frequency, propagating through a plasma.
Suppose that the wave electric field takes the form

$$\vec{E} = E_0 e^{i(kz-\omega t)} \hat{z}$$

We now need to consider how we could transmit information through a plasma (or any other dielectric medium) by means of electromagnetic waves.

The easiest way would be to send a series of short discrete wave-pulses through the plasma, so that we could encode information in a sort of Morse code.
We can build up a wave-pulse from a suitable superposition of infinite plane-waves of different frequencies and wave-lengths: *e.g.*,

$$E_z(x,t)=\int F(k)e^{i\phi(k)}dk$$

$$\phi(k)=kx-\omega t$$

$$\frac{d\phi}{dk}=x-\frac{d\omega}{dk}t=0$$

$$v_g=\frac{d\omega}{dk}$$

The upshot of the above discussion is that information (*i.e.*, an individual wave-pulse) travels through a dispersive media at the group velocity, rather than the phase velocity. Hence, relativity demands that the group velocity, rather than the phase velocity, must always be less than .

What is the group velocity for high frequency waves propagating through a plasma? Well, differentiation of the dispersion relation yields

$$\omega^2 = k^2 c^2 + \omega_p^2$$

$$\rightarrow 2\omega \frac{d\omega}{dk} = 2kc^2$$

$$\frac{\omega}{k}\frac{d\omega}{dk} = v_p v_g = c^2$$

$$v_p = \frac{c}{\sqrt{1-\omega^2{}_p/\omega^2}}$$

$$v_g = c\sqrt{1-\frac{\omega_p{}^2}{\omega^2}}$$

which is less than. We thus conclude that the dispersion relation is indeed consistent with relativity.

Let us now consider the propagation of low frequency electromagnetic waves through a plasma. We can see, from Eqs. , that when the wave frequency, ω , falls below the plasma frequency, ωp , both the phase and group velocities become imaginary.

$$v_p = \frac{c}{\sqrt{1-\omega^2{}_p/\omega^2}}$$

$$v_g = c\sqrt{1-\frac{\omega_p{}^2}{\omega^2}}$$

$$k = \frac{i\sqrt{\omega_p{}^2-\omega^2}}{c} = i\left|k\right|$$

$$E_z = E_0 e^{i(i\left|k\right|x-\omega t)} = E_0 e^{i(i\left|k\right|x} e^{-i\omega t)}$$