Atmospheric Dynamics

Lecture 4

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The Vorticity Equation in Isobaric Coordinates

To obtain a version of the vorticity equation in pressure coordinates, we follow the same procedure as we used to obtain the z-coordinate version:

$$\frac{\partial}{\partial x}\text{ [y-component momentum equation] } - \frac{\partial}{\partial y}\text{ [x-component momentum equation]}$$

Using the p-coordinate form of the momentum equations, this is:

$$\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\omega\frac{\partial v}{\partial p}+fu=-\frac{\partial \Phi}{\partial y}\right]-\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\omega\frac{\partial u}{\partial p}-fv=-\frac{\partial \Phi}{\partial x}\right]$$

$$\frac{d}{dt}\left(\zeta_p+f\right)=-\left(\zeta_p+f\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)_p-\left(\frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p}-\frac{\partial \omega}{\partial x}\frac{\partial v}{\partial p}\right)$$

vorticity equation in isobaric coordinates

Comparing, we see that in the isobaric system there is no vorticity generation by pressure-density solenoids.

$$\frac{d}{dt}\left(\zeta_p + f\right) = -\left(\zeta_p + f\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p - \left(\frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p} - \frac{\partial \omega}{\partial x}\frac{\partial v}{\partial p}\right)$$

$$\frac{d}{dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

This difference arises because in the isobaric system, horizontal partial derivatives are computed with p held constant so that the vertical component of vorticity is

$$\zeta = (\partial v / \partial x - \partial u / \partial y)_p$$

whereas in height coordinates it is $\zeta = (\partial v / \partial x - \partial u / \partial y)_z$

In practice the difference is generally unimportant because, as shown in the next section, the solenoidal term is usually sufficiently small so that it can be neglected for synoptic-scale motions.

The Vorticity Equation

Relates the local rate of change of vorticity to several forcing mechanisms:

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

Divergence Term

Convergence = Increase in vorticity
Divergence = Decrease in vorticity

Tilting Term

Represents effects on vorticity due to changes in the vertical velocity (w) in the horizontal direction

Solenoidal Term

Effects of pressure gradient force on changes in vorticity

Scale Analysis of the Vorticity Equation

The equations of motion were simplified for synoptic-scale motions by evaluating the order of magnitude of various terms.

The same technique can be applied to the vorticity equation.

Characteristic scales for the field variables based on typical observed magnitudes for synoptic-scale motions are chosen as follows:

Values of Scaling Quantities (midlatitude large-scale motions)

U	horizontal velocity scale	$10\ \mathrm{m\ s^{-1}}$
W	vertical velocity scale	$10^{-2}\ \mathrm{m\ s^{-1}}$
L	length scale	$10^{6}\ \mathrm{m}$
H	depth scale	$10^{4}\ \mathrm{m}$
δP	horizontal pressure fluctuation	$10^{3}\ \mathrm{Pa}$
ρ	mean density	$1\ \mathrm{kg\ m^{-1}}$
$\delta\rho/\rho$	fractional density fluctuation	10^{-2}
$T = L/U$	time scale (advective)	$10^{5}\ \mathrm{s}$
f_0	Coriolis parameter	$10^{-4}\ \mathrm{s^{-1}}$
β	= "beta" parameter	$10^{-11}\ \mathrm{m^{-1}\ s^{-1}}$

Scale Analysis of the Vorticity Equation

Starting with the z-coordinate form of the vorticity equation, we begin by expanding the total derivative and retaining only nonzero terms:

$$\frac{d}{dt}(\zeta + f) = \frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} + v\frac{\partial f}{\partial y}$$

$$= -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

Next, we will evaluate the order of magnitude of each of these terms.

Our goal is to simplify the equation by retaining only those terms that are important for large-scale midlatitude weather systems.

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \leq \frac{U}{L} \sim 10^{-5}\, s^{-1}$$

Where the inequality in this expression means less than or equal to in order of magnitude. Thus,

$$\frac{\zeta}{f_0} \leq \frac{U}{f_0 L} \equiv R_o \sim 10^{-1}$$

For midlatitude synoptic-scale systems, the relative vorticity is often small (order Rossby number) compared to the planetary vorticity.

For such systems, ξ may be neglected compared to f in the divergence term in the vorticity equation

The Rossby Number

The Rossby Number is a dimensionaless number used in describing geophysical phenomena in the oceans and atmosphere

It characterises the ratio of inertial forces in a fluid to the fictitious forces arising from planetary rotation

$$R_o \equiv \frac{U^2 / L}{f_0 U} = \frac{U}{f_0 L}$$

GRADIENT BALANCE (ROSSBY NUMBER NEAR UNITY)

- If the Rossby number is of the order of unity ($Ro \sim 1$), then all three terms must be retained. This is known as *gradient balance*, and the wind in this case is known as the *gradient wind*. Details of gradient wind will be discussed in a future lesson.
- The following table summarized these results

Ro	Terms	Balance
$\ll 1$	pressure gradient and Coriolis	*geostrophic*
~ 1	acceleration, pressure gradient, and Coriolis	*gradient*
$\gg 1$	acceleration and pressure gradient	*cyclostrophic*

- For large-scale (synoptic scale) motion, the Rossby number is of the order

$$Ro \sim \frac{(10 m/s)}{(10^{-4} s^{-1})(10^{6} m)} = 0.1,$$

which shows that on these scales the atmosphere is close to being in geostrophic balance. Hence, the actual wind should be close to the geostrophic wind.

$$(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \sim f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \leq \frac{f_0 U}{L} \sim 10^{-9}\ s^{-2}$$

Since both sides of the equation must balance, this term must be smaller, which implies that large scale motions are quasi-nondivergent.

This approximation does not apply near the center of intense cyclonic storms.

In such systems $|\zeta / f| \sim 1$ and the relative vorticity should be retained.

The magnitudes of the various terms in equation can now be estimated as

$$\frac{\partial \zeta}{\partial t}, u\frac{\partial \zeta}{\partial x}, v\frac{\partial \zeta}{\partial y} \sim \frac{U^2}{L^2} \sim 10^{-10}\ s^{-2}$$

$$w\frac{\partial \zeta}{\partial z} \sim \frac{WU}{HL} \sim 10^{-11}\ s^{-2}$$

$$v\frac{\partial f}{\partial y} \sim U\beta \sim 10^{-10}\ s^{-2}$$

$$\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) \leq \frac{WU}{HL} \sim 10^{-11}\ s^{-2}$$

These inequalities appear because the two terms may partially offset one another

$$\frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial\rho}{\partial y}\frac{\partial p}{\partial x}\right) \le \frac{\delta\rho\,\delta p}{\rho^2 L^2} \quad \sim 10^{-11}\ s^{-2}$$

These inequalities appear because the two terms may partially offset one another.

$\delta\rho/\rho$	*fractional density fluctuation*	10^{-2}

$\delta P/\rho \sim (10\ \mathrm{hPa}/1) \sim (10^3\ \mathrm{m^2 s^{-2}})$ pressure horizontal variation scale

$$\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}+\cancel{w\frac{\partial \zeta}{\partial z}}+v\frac{\partial f}{\partial y}=-\,\zeta+f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\cancel{\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)}+\cancel{\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)}$$

10^{-10} 10^{-11} 10^{-10} 10^{-9} 10^{-11} 10^{-11}

$$\frac{df}{dt}=\cancel{\frac{\partial f}{\partial t}}+\cancel{u\frac{\partial f}{\partial x}}+v\frac{\partial f}{\partial y}+\cancel{w\frac{\partial f}{\partial z}}$$

$$\underbrace{\frac{\partial(\zeta+f)}{\partial t}}_{\text{local tendency of absolute vorticity}}+\underbrace{u\frac{\partial(\zeta+f)}{\partial x}+v\frac{\partial(\zeta+f)}{\partial y}}_{\text{horizontal advection of absolute vorticity}}=\underbrace{-(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}_{\text{effects of divergence}}$$

Implications: For large-scale weather systems, the change of absolute vorticity following the fluid motion is approximately equal to the production of vorticity due to horizontal convergence or the destruction of vorticity due to horizontal divergence.

(Note that this does not hold at smaller scales, for which the vertical advection, tilting and baroclinic terms may also be important)