Atmospheric Dynamic

Lecture 2

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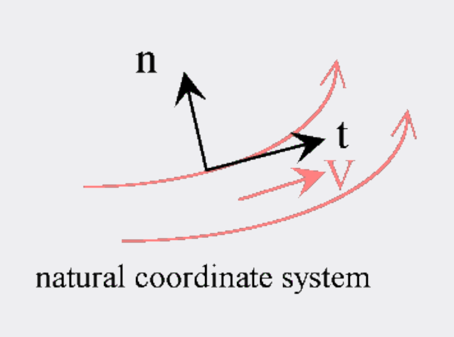
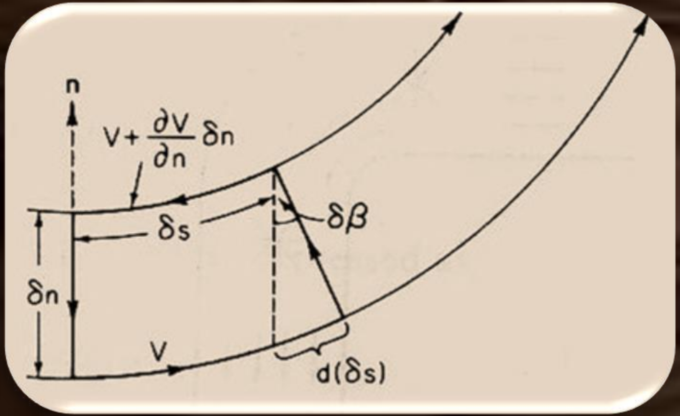
Vorticity in Natural Coordinates



Using natural coordinates can make it easier to physically interpret the relationship between relative vorticity and the flow.

To express vorticity in natural coordinates, we compute the circulation around the infinitesimal contour shown below.



$$\delta C = V\left[\delta s + d\left(\delta s\right)\right] - \left(V + \frac{\partial V}{\partial n}\delta n\right)\delta s$$

From the diagram, $d(\delta s) = \delta\beta\, \delta n$, where $\delta\beta$ is the angular change in wind direction in the distance δn.

$$\delta C = \left(-\frac{\partial V}{\partial n} + V\frac{\partial \beta}{\partial s}\right)\delta n \delta s$$

in the limit $\delta n \delta s \to 0$

$$\zeta = \lim_{\delta n, \delta s \to 0} \frac{\delta C}{\delta n \delta s} = -\frac{\partial V}{\partial n} + \frac{V}{R_s}$$

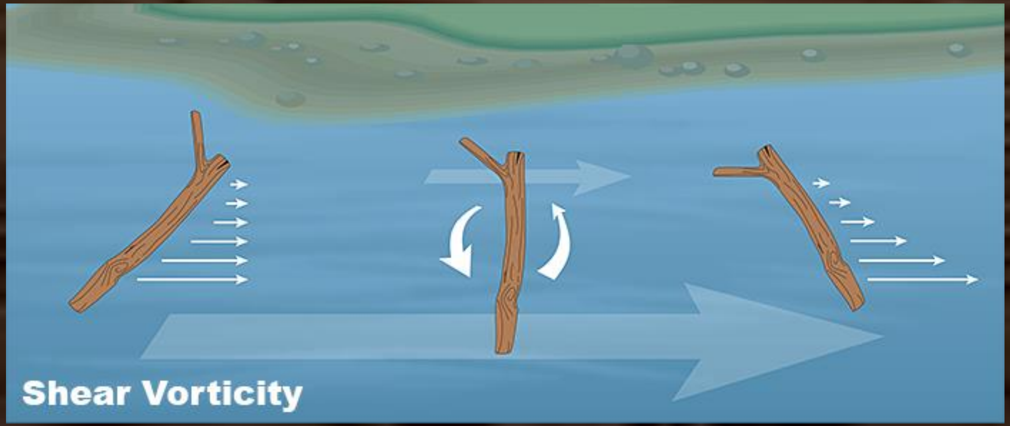
where R_s is the radius of curvature of the streamlines

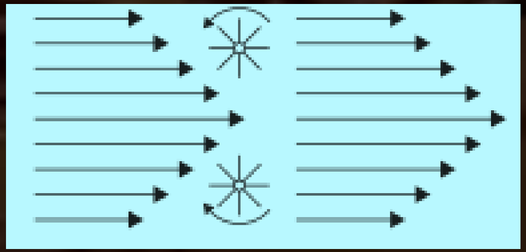
Vorticity in natural coordinates: $$\zeta = -\frac{\partial V}{\partial n} + \frac{V}{R_s}$$

It is now apparent that the net vertical vorticity component is the result of the sum of two parts:

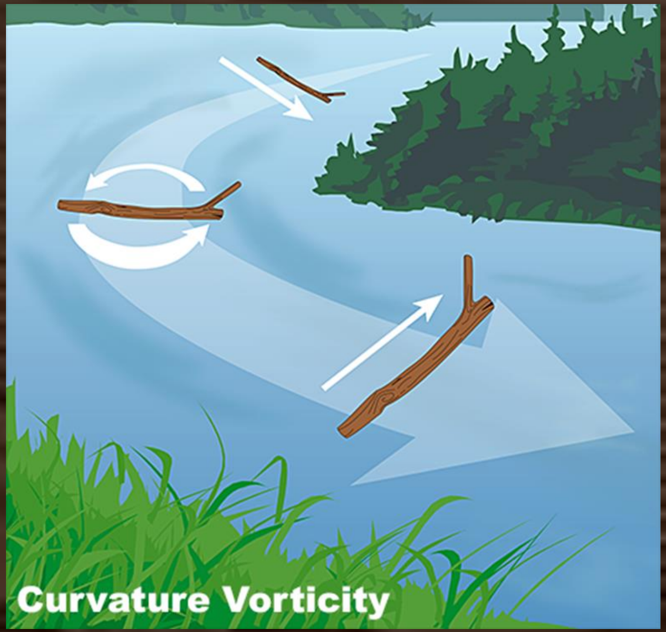
$$-\frac{\partial V}{\partial n}$$

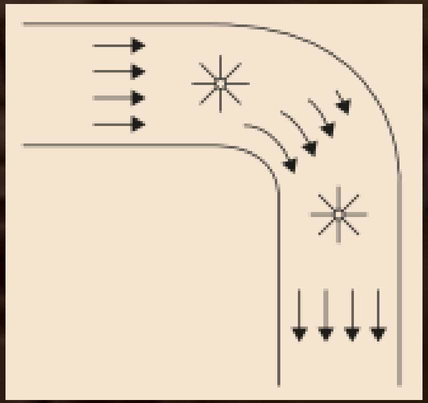
The rate of change of wind speed normal to the direction of flow, which is called the shear vorticity.

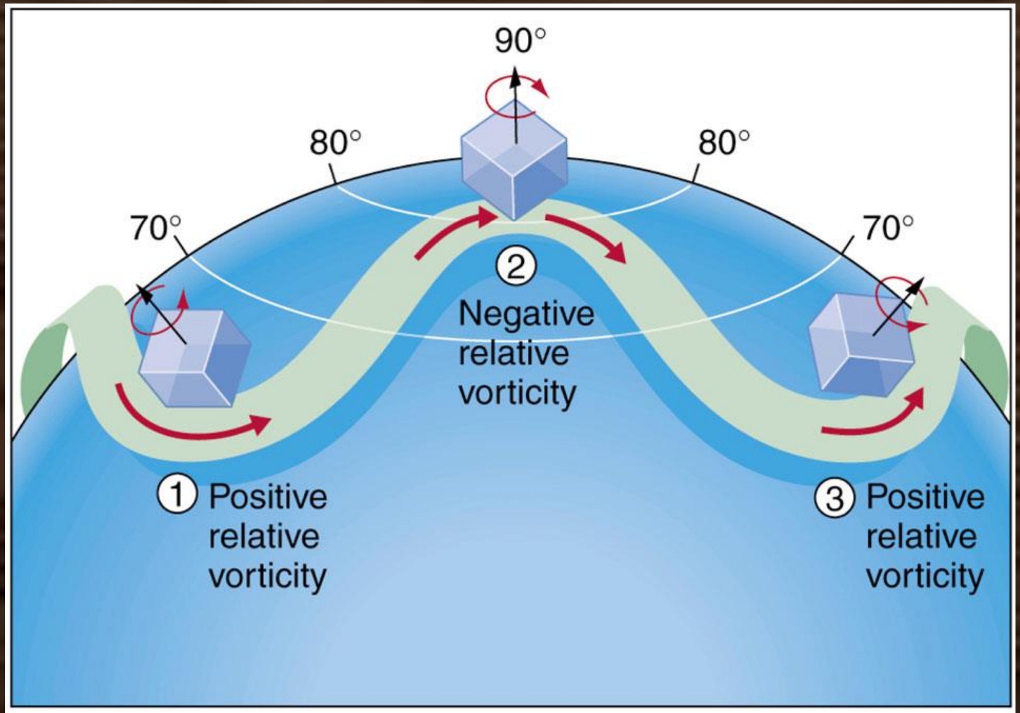


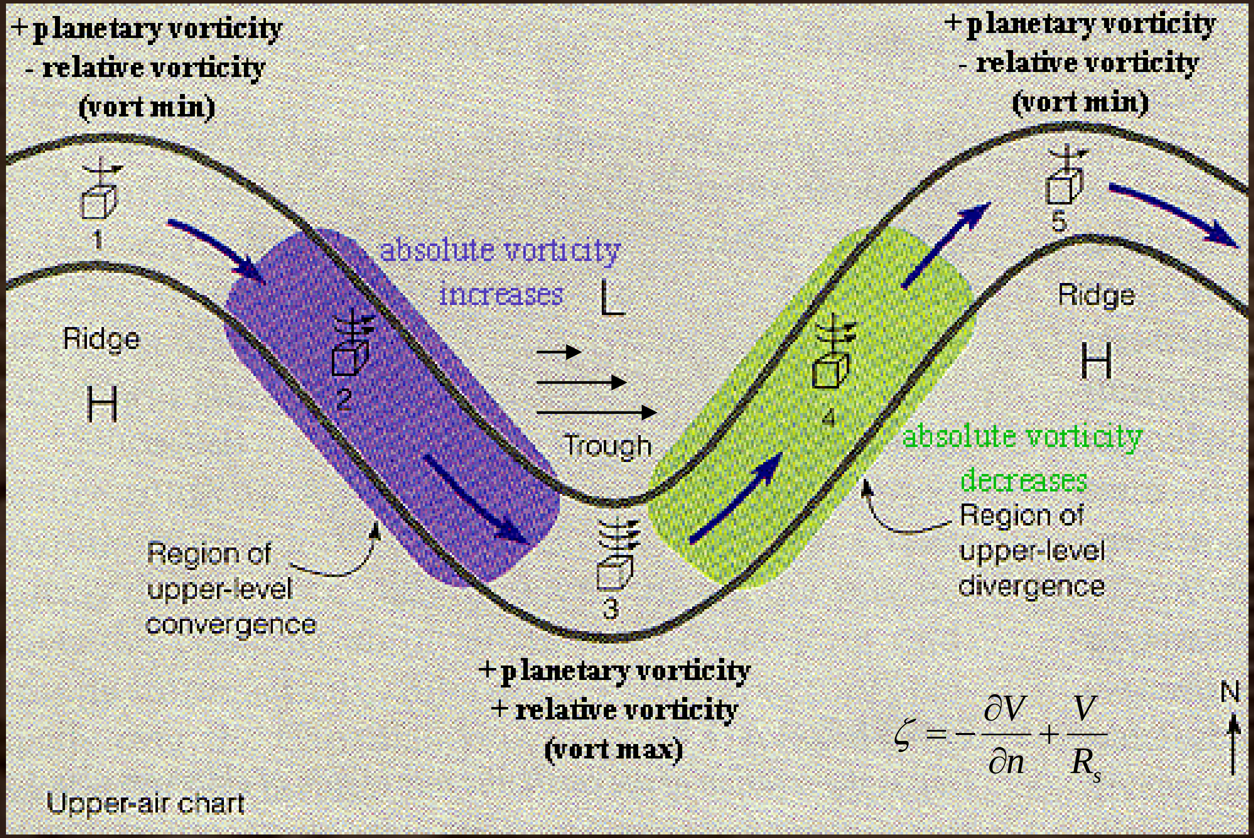


$\frac{V}{R_s}$ The turning of the wind along a streamline, which is called the curvature vorticity.



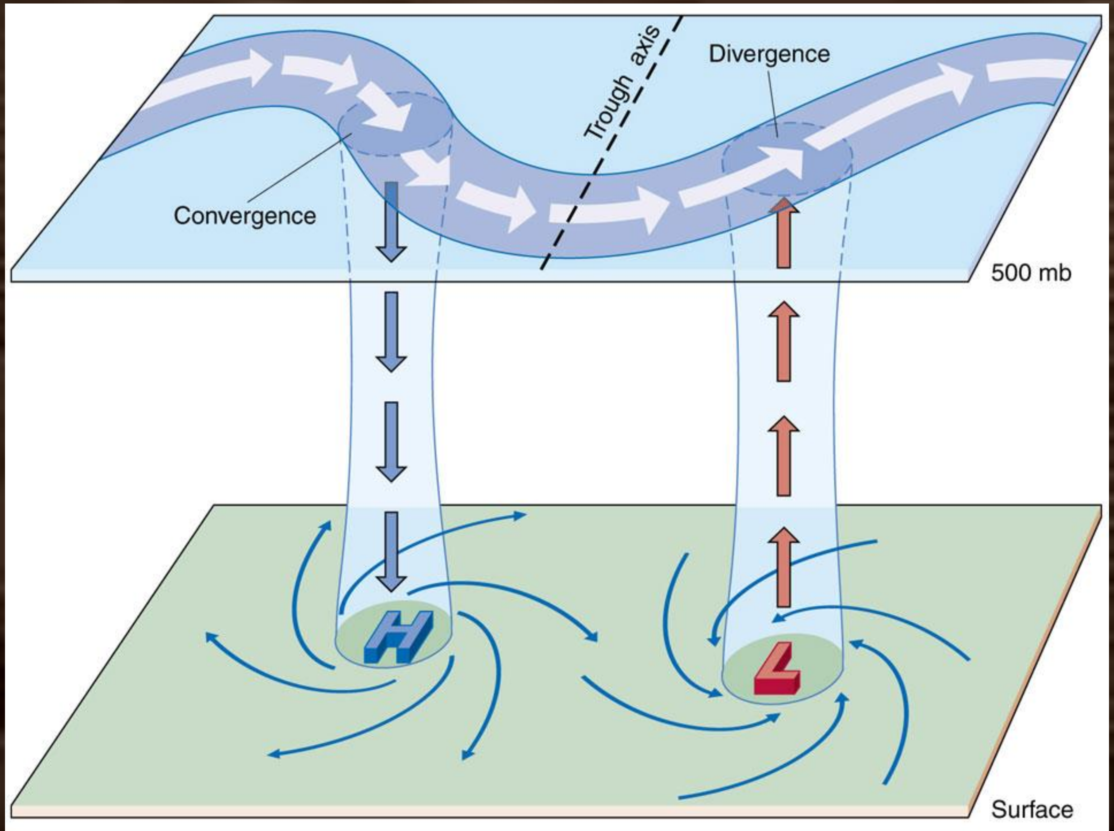


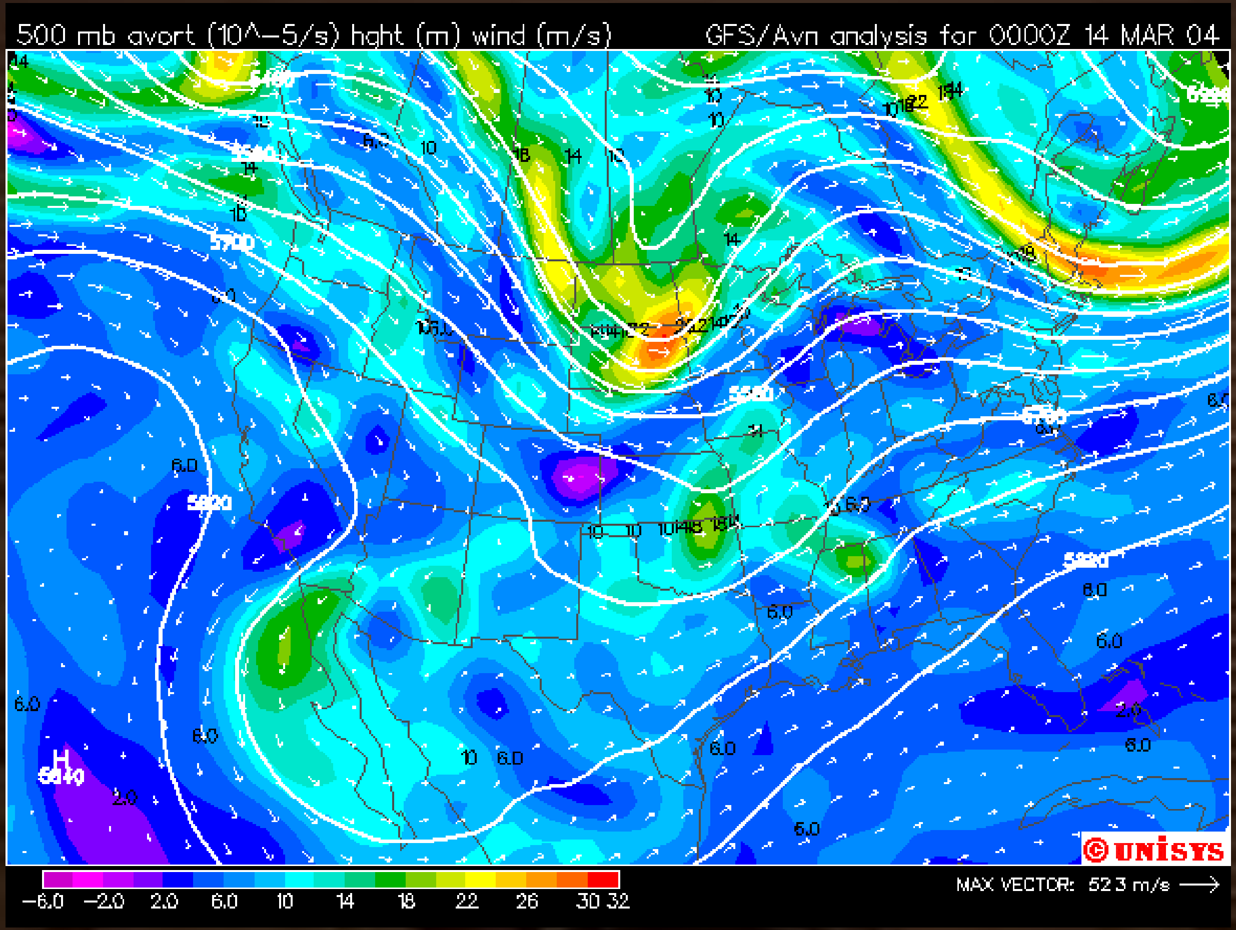

90°
80°
80°
70°
70°
2
Negative relative vorticity
1
Positive relative vorticity
3
Positive relative vorticity



Vorticity Maximum: Along the trough axis to left of the strongest flow. Both shear and curvature terms are positive.

Vorticity Minimum: Along the ridge axis to right of the strongest flow. Both shear and curvature terms are negative.

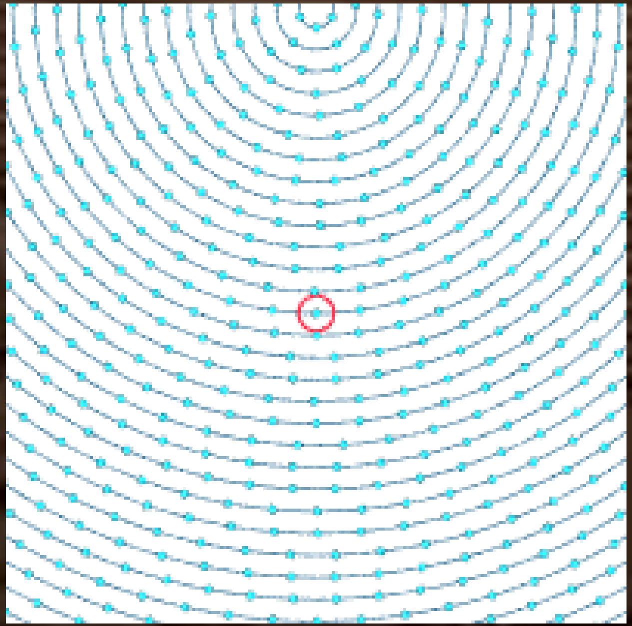

Trough axis
Divergence
Convergence
500 mb
Surface



Absolute vorticity = $\zeta + f$

In a mass of continuum that is rotating like a rigid body, the vorticity is twice the angular velocity vector of that rotation.

This is the case, for example, of water in a tank that has been spinning for a while around its vertical axis, at a constant rate.



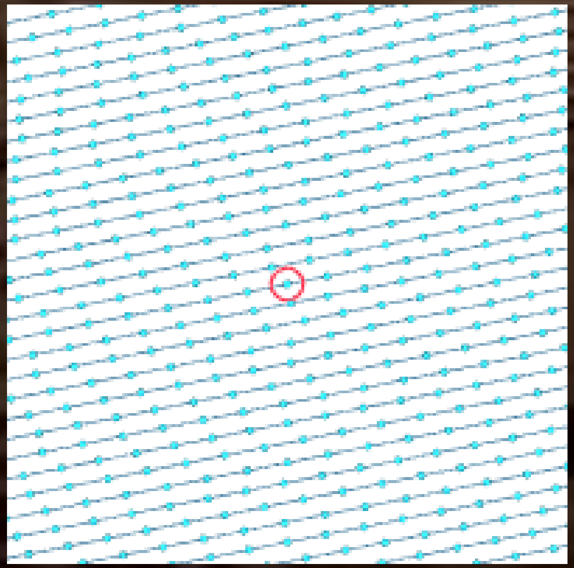
Rigid-body-like vortex

$v \propto r$

The vorticity may be nonzero even when all particles are flowing along straight and parallel pathlines, if there is shear (that is, if the flow speed varies across streamlines).

For example, in the laminar flow within a pipe with constant cross section all particles travel parallel to the axis of the pipe; but faster near that axis, and practically stationary next to the walls.

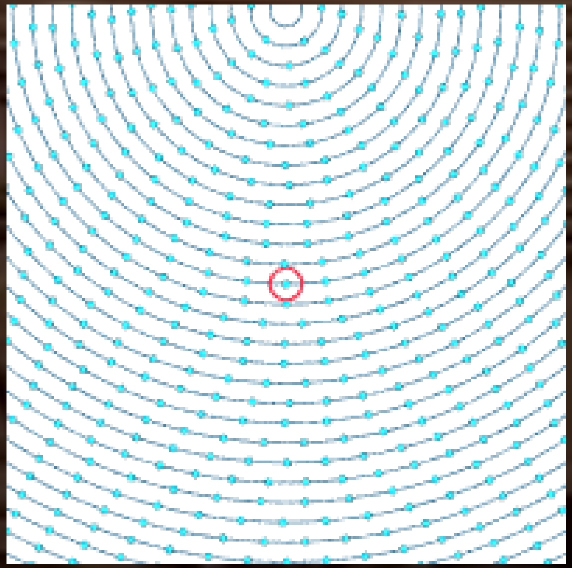
The vorticity will be zero on the axis, and maximum near the walls, where the shear is largest.



Parallel flow with shear

Conversely, a flow may have zero vorticity even though its particles travel along curved trajectories.

An example is the ideal irrotational vortex, where most particles rotate about some straight axis, with speed inversely proportional to their distances to that axis. A small parcel of continuum that does not straddle the axis will be rotated in one sense but sheared in the opposite sense, in such a way that their mean angular velocity *about their center of mass* is zero.



Irrotational vortex
$v \propto 1/r$

The Vorticity Equation

The previous section discussed kinematic properties of vorticity.

This section addresses vorticity dynamics using the equations of motion to determine contributions to the time rate of change of vorticity.

Cartesian Coordinate Form

For motions of synoptic scale, the vorticity equation can be derived using the approximate horizontal momentum equations.

We differentiate the zonal component equation with respect to *y* and the meridional *componentequation with respect to x:*

Vector momentum equation in rotating coordinates

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

Rate of change of relative velocity following the relative motion in a rotating reference frame

Coriolis acceleration

Pressure gradient force (per unit mass)

Gravity term (gravitation + centrifugal)

Friction

$$\frac{du}{dt} = \ldots$$ x-component momentum equation

$$\frac{dv}{dt} = \ldots$$ y-component momentum equation

Above the boundary layer, all horizontal parcel accelertions can be understood by comparing the magnitude and direction of the pressure gradient and coriolis forces.

$$\begin{cases} \dfrac{du}{dt} = -\dfrac{1}{\rho}\dfrac{\partial P}{\partial x} + fv \\ \dfrac{dv}{dt} = -\dfrac{1}{\rho}\dfrac{\partial P}{\partial y} - fu \end{cases}$$

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z} - fv = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z} + fu = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} \end{cases}$$

$$\frac{\partial}{\partial x}\text{[y-component momentum equation]}$$

$$-\frac{\partial}{\partial y}\text{[x-component momentum equation]} =$$

$$\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu=-\frac{1}{\rho}\frac{\partial p}{\partial y}\right]$$

$$-\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x}\right]$$