Atmospheric Dynamics

Lecture 15

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The Quasi-Geostrophic Vorticity Equation

Just as the horizontal momentum can be approximated to O(Ro) by its geostrophic value, the vertical component of vorticity can also be approximated geostrophically.

In Cartesian coordinates the components of $\mathbf{V}_g \equiv f_0^{-1}\mathbf{k} \times \nabla\Phi$ are

$$f_0 v_g = \frac{\partial \Phi}{\partial x}, \qquad f_0 u_g = -\frac{\partial \Phi}{\partial y}$$

Thus, the geostrophic vorticity, $\zeta_g = \mathbf{k} \cdot \nabla \times \mathbf{V}_g$, , can be expressed as

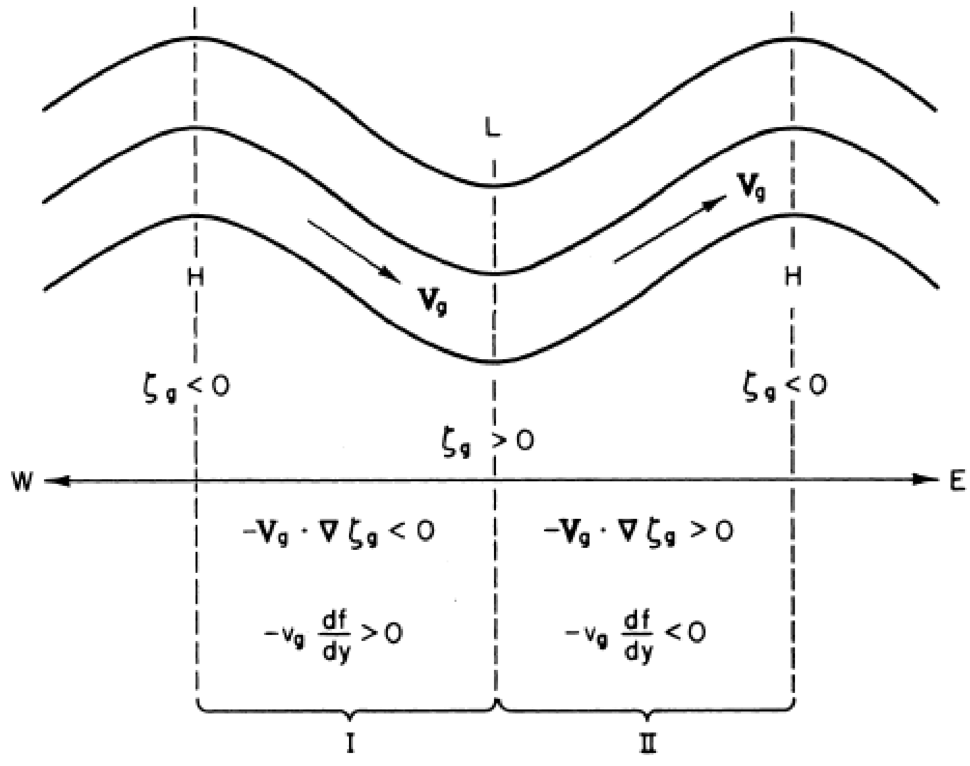
$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) = \frac{1}{f_0}\nabla^2\Phi$$

This *invertibility* is one reason why vorticity is such a useful forecast diagnostic; if the evolution of the vorticity can be predicted, then inversion of this evolution yields the evolution of the geopotential field, from which it is possible to determine the geostrophic wind and temperature distributions.

Since the Laplacian of a function tends to be a maximum where the function itself is a minimum, positive vorticity implies low values of geopotential and vice versa, as illustrated for a simple sinusoidal disturbance in Fig.

Schematic 500-hPa geopotential field showing regions of positive and negative advections of relative and planetary vorticity.

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) = \frac{1}{f_0}\nabla^2\Phi$$



The quasi-geostrophic vorticity equation can be obtained from the x and y components of the quasi-geostrophic momentum equation,

$$\frac{D_g \mathbf{V}_g}{Dt} = -f_0 \mathbf{k} \times \mathbf{V}_a - \beta y \mathbf{k} \times \mathbf{V}_g$$

which can be expressed, respectively, as

$$\frac{D_g u_g}{Dt} - f_0 v_a - \beta y v_g = 0$$

$$\frac{D_g v_g}{Dt} + f_0 u_a + \beta y u_g = 0$$

$$\mathbf{V}_g \equiv f_0^{-1} \mathbf{k} \times \nabla \Phi$$

$$\frac{D_g \zeta_g}{Dt} = -f_0 \left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} \right) - \beta v_g$$

which should be compared with

$$\frac{D_h (\zeta + f)}{Dt} = -(\zeta + f) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

Noting that because f depends only on y so that $D_g f/Dt = \mathbf{V}_g \cdot \nabla f = \beta v_g$

and that the divergence of the ageostrophic wind can be eliminated in favor of ω using

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

we can rewrite $\frac{D_g \zeta_g}{Dt} = -f_0 \left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} \right) - \beta v_g$ as

$$\frac{\partial \zeta_g}{\partial t} = -\mathbf{V}_g \cdot \nabla (\zeta_g + f) + f_0 \frac{\partial \omega}{\partial p} \quad *$$

which states that the local rate of change of geostrophic vorticity is given by the sum of the advection of the absolute vorticity by the geostrophic wind plus the concentration or dilution of vorticity by stretching or shrinking of fluid columns (the divergence effect).

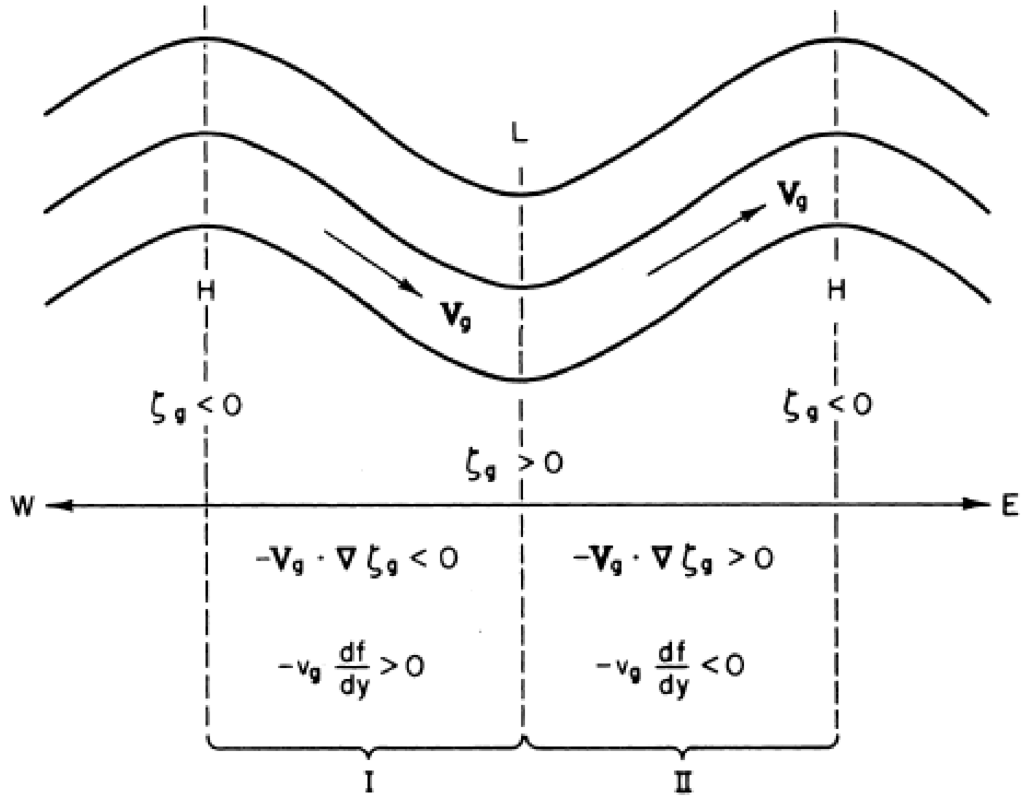
The vorticity tendency due to vorticity advection [the first term on the right in (*)] may be rewritten as

$$-\mathbf{V}_g \cdot \nabla (\zeta_g + f) = -\mathbf{V}_g \cdot \nabla \zeta_g - \beta v_g$$

The two terms on the right represent the geostrophic advections of relative vorticity and planetary vorticity, respectively. For disturbances in the westerlies, these two effects tend to have opposite signs, as illustrated schematically in Fig. for an idealized 500-hPa flow.

In region I upstream of the 500-hPa trough, the geostrophic wind is directed from the relative vorticity minimum at the ridge toward the relative vorticity maximum at the trough so that

$$-V_g \bullet \nabla \zeta_g < 0$$



we consider an idealized geopotential distribution on a midlatitude β-plane consisting of the sum of a zonally averaged part, which depends linearly on y, and a zonally varying part (representing a synoptic wave disturbance) that has a sinusoidal dependence in x and y:

$$\Phi(x, y) = \Phi_0 - f_0 U y + f_0 A \sin kx \cos ly$$

Here, $y = a(\phi - \phi_0)$, where a is the radius of the earth and ϕ_0 is the latitude at which f_0 is evaluated.

The paramers Φ_0, U, and A depend only on pressure,

the *wave numbers* k and l are defined as $k = 2\pi/L_x$ and $l = 2\pi/L_y$ with L_x and L_y the wavelengths in the x and y directions, respectively.

The geostrophic wind components are then given by

$$u_g = -\frac{1}{f_0}\frac{\partial \Phi}{\partial y} = U + u_g' = U + lA \sin kx \sin ly$$
$$v_g = \frac{1}{f_0}\frac{\partial \Phi}{\partial x} = v_g' = +kA \cos kx \cos ly$$

The geostrophic vorticity is then simply

$$\zeta_g = f_0^{-1}\nabla^2\Phi = -\left(k^2 + l^2\right) A \sin kx \, \cos ly$$

With the aid of these relations it can be shown that in this simple case the advection of relative vorticity by the wave component of the geostrophic wind vanishes:

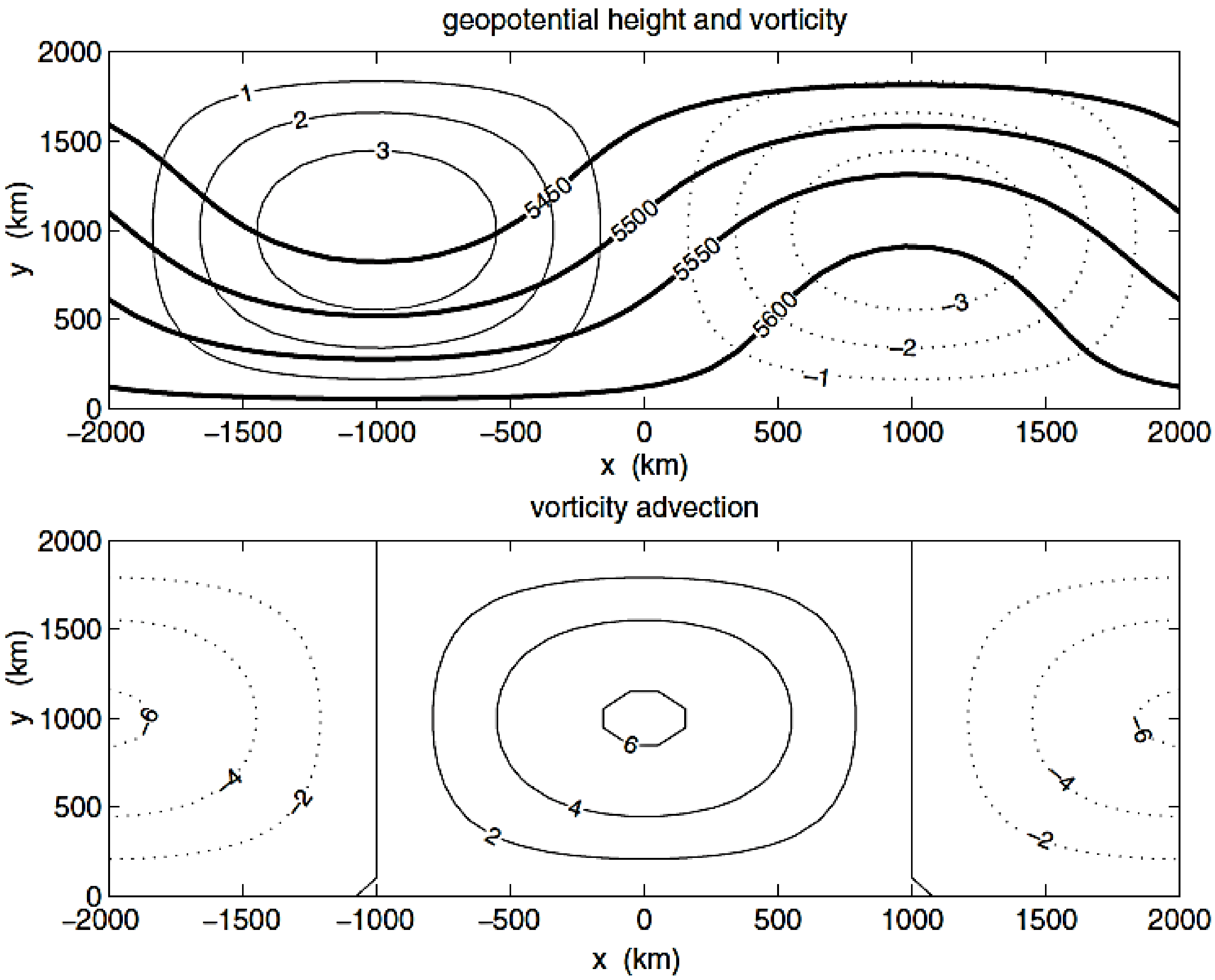
$$u_g' \partial \zeta_g \big/ \partial x + v_g' \partial \zeta_g \big/ \partial y = 0$$

so that the advection of relative vorticity is simply

$$-u_g \frac{\partial \zeta_g}{\partial x} - v_g \frac{\partial \zeta_g}{\partial y} = -U \frac{\partial \zeta_g}{\partial x} = +kU\left(k^2 + l^2\right) A \cos kx \cos l y$$

and the advection of planetary vorticity can be expressed as

$$-\beta v_g = -\beta k A \cos kx \cos l y$$


geopotential height and vorticity
y (km)
x (km)
5450
5500
5550
5600
1
2
3
−1
−2
−3
vorticity advection
y (km)
x (km)
2
4
6
−2
−4
−6

QUASI-GEOSTROPHIC PREDICTION

Although, as explained above, the ageostrophic vertical motion plays an essential role in the maintenance of thermal wind balance as the flow evolves, the evolution of the geostrophic circulation can actually be determined without explicitly determining the distribution of ω.

Defining the geopotential tendency $\chi \equiv \partial\Phi/\partial t$ and recalling that the order of partial differentiation may be reversed, the geostrophic vorticity equation

$$\frac{\partial \zeta_g}{\partial t} = -\mathbf{V}_g \cdot \nabla \left(\zeta_g + f\right) + f_0 \frac{\partial \omega}{\partial p}$$

can be expressed as

$$\frac{1}{f_0}\nabla^2 \chi = -\mathbf{V}_g \cdot \nabla \left(\frac{1}{f_0}\nabla^2 \Phi + f\right) + f_0 \frac{\partial \omega}{\partial p}$$

where we have used

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) = \frac{1}{f_0}\nabla^2 \Phi$$

to write the geostrophic vorticity and its tendency in terms of the Laplacian of geopotential.

Thus, since by

$$\mathbf{V}_g \equiv f_0^{-1}\mathbf{k}\times\nabla\Phi$$

the geostrophic wind can be expressed in terms of , the right-hand side of Φ

$$\frac{1}{f_0}\nabla^2\chi = -\mathbf{V}_g\cdot\nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right) + f_0\frac{\partial\omega}{\partial p}$$

depends on the dependent variables Φ and ω alone.

An analogous equation dependent on these two variables can be obtained from the thermodynamic energy equation

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_g\cdot\nabla\right)\left(-\frac{\partial\Phi}{\partial p}\right) - \sigma\omega = \frac{\kappa J}{p}$$

by multiplying through by f_0/σ and differentiating with respect to p. Using the definition of χ given above, the result can be expressed as

$$\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial\chi}{\partial p}\right) = -\frac{\partial}{\partial p}\left[\frac{f_0}{\sigma}\mathbf{V}_g\cdot\nabla\left(\frac{\partial\Phi}{\partial p}\right)\right] - f_0\frac{\partial\omega}{\partial p} - f_0\frac{\partial}{\partial p}\left(\frac{\kappa J}{\sigma p}\right)$$

where σ was defined below $$\left(\frac{\partial}{\partial t} + \mathbf{V}_g \bullet \boldsymbol{\nabla}\right) T - \left(\frac{\sigma p}{R}\right)\omega = \frac{J}{c_p}$$

The ageostrophic vertical motion, ω, has equal and opposite effects on the left hand sides in

$$\frac{1}{f_0}\boldsymbol{\nabla}^2\chi = -\mathbf{V}_g \bullet \boldsymbol{\nabla}\left(\frac{1}{f_0}\boldsymbol{\nabla}^2\Phi + f\right) + f_0\frac{\partial \omega}{\partial p} \quad *$$

and.

$$\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \chi}{\partial p}\right) = -\frac{\partial}{\partial p}\left[\frac{f_0}{\sigma}\mathbf{V}_g \bullet \boldsymbol{\nabla}\left(\frac{\partial \Phi}{\partial p}\right)\right] - f_0\frac{\partial \omega}{\partial p} - f_0\frac{\partial}{\partial p}\left(\frac{\kappa J}{\sigma p}\right) \quad **$$

Vertical stretching ($\partial\omega/\partial p > 0$) forces a positive tendency in the geostrophic vorticity (*) and a negative tendency of equal magnitude in the term on the left side in (**).

$$\frac{\partial \Phi}{\partial p} = -\alpha = -RT/p$$

$$\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \chi}{\partial p}\right) = -Rf_0\frac{\partial}{\partial p}\left(\frac{1}{\sigma p}\frac{\partial T}{\partial t}\right) = -f_0\frac{\partial}{\partial p}\left(\frac{1}{S_p}\frac{\partial T}{\partial t}\right) \approx -\frac{\partial}{\partial t}\left(\frac{f_0}{S_p}\frac{\partial T}{\partial p}\right)$$

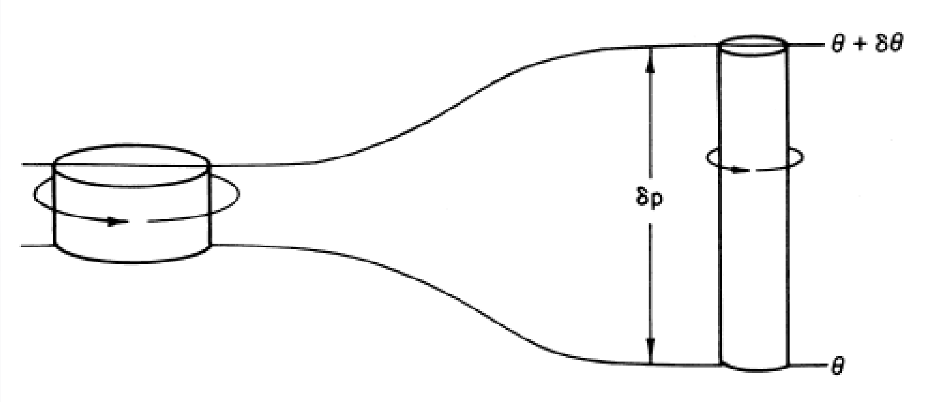
where we have used the fact that $S_p = p\sigma / R$ varies only slowly with height in the troposphere. Because T is the departure of temperature from its standard atmosphere value, the expression

$$f_0 S_p^{-1} \partial T / \partial p$$

is proportional to the local static stability anomaly divided by the standard atmosphere static stability. Multiplication by f_0 gives this expression the same units as vorticity.

$$(\partial\omega/\partial p > 0)$$

As was shown in Fig.



an air column that moves adiabatically from a region of high static stability to a region of low static stability is stretched vertically ($\partial\omega/\partial p > 0$) so that the upper portion of the column cools adiabatically relative to the lower portion.

Thus, the relative vorticity in (*) and the normalized static stability anomaly in (**) are changed by equal and opposite amounts.

For this reason the normalized static stability anomaly is referred to as the *stretching vorticity*.

Purely geostrophic motion ($\omega = 0$) is a solution to (*) and (**) only in very special situations such as barotropic flow (no pressure dependence) or zonallysymmetric flow (no x dependence).

More general purely geostrophic flows cannot satisfy both these equations simultaneously, as there are then two independent equations in a single unknown so that the system is over-determined.

Thus, it should be clear that the role of the vertical motion distribution must be to maintain consistency between the geopotential tendencies required by vorticity advection in (*) and thermal advection in (**).